TOPIC 2.5- Covariance

Covariance

If X and Y are random variables, then covariance between X and Y is defined as $\text{cov}(X,Y) = E[XY] - E[X]\,E[Y]$

If X and Y are independent, then

$$E[XY] = E[X]\,E[Y]$$

$$\Rightarrow \boxed{\text{cov}(X,Y) = 0}$$

Find the covariance of x and y if the random variable (X,Y) has the joint pdf $f(x,y) = x+y$, $0 \le x \le 1, ~~y \le 1~~$ $0 \le y \le 1$.

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_0^1$$

$$= \left[x + \frac{1}{2}\right], \quad 0 \le x \le 1$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (x+y)\,dx = \left[\frac{x^2}{2} + xy\right]_0^1$$

$$= \left[y + \frac{1}{2}\right], \quad 0 \le y \le 1$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\,dx = \int_0^1 x\left(x + \frac{1}{2}\right)dx$$

$$= \int_0^1 \left[x^2 + \frac{x}{2}\right]dx = \left[\frac{x^3}{3} + \frac{x^2}{4}\right]_0^1 = \left[\frac{1}{3} + \frac{1}{4}\right]$$

$$= 7/12$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f(y)\,dy = \int_0^1 y\left(y + \frac{1}{2}\right)dy$$

$$= \int_0^1 \left[y^2 + \frac{y}{2}\right]dy = \left[\frac{y^3}{3} + \frac{y^2}{4}\right]_0^1 = \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{12}$$

$$E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^1 xy\,(x+y)\, dx\, dy$$

$$= \int_0^1\int_0^1 \left[x^2y + xy^2\right] dx\, dy$$

$$= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy$$

$$= \int_0^1 \left[\frac{y}{3} + \frac{y^2}{2}\right] dy = \left[\frac{y^2}{6} + \frac{y^3}{6}\right]_0^1$$

$$= \left[\frac{1}{6} + \frac{1}{6}\right] = \frac{1}{3}$$

$$Cov(x,y) = E[xy] - E[x]\,E[y]$$

$$= \frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = \frac{1}{3} - \frac{49}{144}$$

$$= \frac{48-49}{144} = -\frac{1}{144}$$

(2) The joint probability density function of the random variable x & y is defined as

$$f(x,y) = \begin{cases} 25e^{-5y}, & 0<x<2,\ y>0 \\ 0, & \text{otherwise} \end{cases}$$

Find the covariance of x and y.

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} 25e^{-5y}\,dy$$

$$= 25\left[\frac{e^{-5y}}{-5}\right]_0^{\infty} = -5\left[0-1\right] = 5,\quad 0<x<2$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{2} 25e^{-5y}\,dx$$

$$= 25e^{-5y}\left(x\right)_0^2 = 50e^{-5y},\quad y>0$$

$$E[X] = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_0^2 5x\,dx = 5\left(\frac{x^2}{2}\right)_0^2$$

$$= 10$$

$$E[Y] = \int_{-\infty}^{\infty} y\,f(y)\,dy = \int_0^{\infty} y\,50e^{-5y}\,dy$$

$$= 50\left[y\left(\frac{e^{-5y}}{-5}\right) - \left(\frac{e^{-5y}}{25}\right)\right]_0^{\infty}$$

$$= 50\left[(0) - \left(0 - \frac{1}{25}\right)\right] = 2$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_{0}^{2}\int_{0}^{\infty} xy\, 25 e^{-5y}\, dy\, dx$$

$$= 25 \int_{0}^{2} x\, dx \int_{0}^{\infty} y e^{-5y}\, dy$$

$$= 25 \left(\frac{x^2}{2}\right)_0^2 \left[y\left(\frac{e^{-5y}}{-5}\right) - \left(\frac{e^{-5y}}{25}\right)\right]_0^\infty$$

$$= 50\left[(0) - \left(0 - \frac{1}{25}\right)\right] = 2$$

$$\text{Cov}(x,y) = E[XY] - E[X]\,E[Y]$$

$$= 2 - (10)(2) = -18$$