TOPIC : 1.6 Moments-Moment Generating function

Moments :

Mathematical Expectation :

Let X be a random variable with pdf (or pmf) $f(x)$. Then its mathematical expectation, denoted by $E(X)$ is given by

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx \quad [\text{for continuous R.V}]$$

$$E(X) = \sum x f(x) \quad [\text{for discrete R.V}]$$

Moments : [Discrete case]

Let X be discrete R.V ~~taking~~ with pmf $p(x)$ then rth moment about the origin is

$$\mu_r' \text{ (about the origin) (or) raw moments} = \sum x^r p(x)$$

and $$\mu_r' \text{ (about any point } x = A) = \sum (x-A)^r p(x)$$

& $$\left.\begin{array}{l}\mu_r \text{ (about mean)} \\ \text{(or) [Central moments]}\end{array}\right\} = \sum (x - \text{mean})^r p(x)$$

$[\text{mean} = \bar{x}]$

Moments [continuous case]

If X is a continuous RV with pdf $f(x)$ then defined in the interval (a,b)

$$\mu_r' \text{ (about origin) (or) raw moments} = \int_a^b x^r f(x)\, dx$$

$$\mu_r' \text{ (about a point } A) = \int_a^b (x-A)^r f(x)\, dx$$

$$\mu_r \text{ (about the mean) (or) (Central moments)} = \int_a^b (x-\bar{x})^r f(x)\, dx$$

Relation between μ_r' & μ_r

$\mu_1' = \text{mean } \bar{x}$

$\mu_2 = \mu_2' - (\mu_1')^2$ [variance]

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

Note:

(1) $\mu_1 = 0$ (always)

(2) $E[ax+b] = aE(x) + b$

$E(ax+b) = E(ax) + E(b)$
$= aE(x) + b$

(3) $E[\varphi(x)+a] = E[\varphi(x)] + a$

(4) $Var(ax+b) = a^2 Var(x)$

(5) $Var(x+k) = Var(x)$

Problems based on Moments [Discrete Case]

1. The monthly demand for Allwyn watches is known to have the following probability distribution.

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Find the expected demand for watches. Also compute the variance.

Soln:

x_i	1	2	3	4	5	6	7	8
$P(x_i)$	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Let x be the R.v denoting the monthly demand for Allwyn watches.

$$E(x) = \sum_{i=1}^{8} x_i p(x_i)$$

$$= 1(0.08) + 2(0.12) + 3(0.19) + 4(0.24) + 5(0.16) + 6(0.10) + 7(0.07) + 8(0.04)$$

$$= 4.06$$

$$E(x^2) = \sum_{i=1}^{8} x_i^2 p(x_i)$$

$$= 1(0.08) + 4(0.12) + 9(0.19) + 16(0.24) + 25(0.16) + 36(0.10) + 49(0.07) + 64(0.04)$$

$$= 19.7$$

$$Var(x) = E(x^2) - [E(x)]^2$$

$$= 19.7 - (4.06)^2$$

$$= 19.7 - 16.48$$

$$= 3.22$$

Problems based on Moments [continuous case]

1. The density function of a random variable 'x' is given by $f(x) = kx(2-x)$, $0 \le x \le 2$. Find k, mean, variance and r^{th} moment.

Soln: Given $f(x) = kx(2-x)$, $0 \le x \le 2$

To find k:

$$\int f(x)\,dx = 1$$

$$\int_0^2 kx(2-x)\,dx = 1$$

$$k\int_0^2 (2x - x^2)\,dx = 1$$

$$k\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[(4 - 8/3) - (0-0)\right] = 1$$

$$k\left[\frac{12-8}{3}\right] = 1$$

$$\Rightarrow k = 3/4$$

To Find r^{th} moment: $E[x^r]$

$$E[x^r] = \int_0^2 x^r f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^r(2x - x^2)\,dx$$

$$= \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})\,dx$$

$$= \frac{3}{4}\left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2$$

$$= \frac{3}{4}\left[\left(2\frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3}\right) - (0-0)\right]$$

$$= \frac{3}{4}\left[\frac{2^{r+3}}{r+2} - \frac{2^{r+3}}{r+3}\right]$$

$$= \frac{3}{4} \cdot 2^{r+3}\left[\frac{1}{r+2} - \frac{1}{r+3}\right]$$

$$= \frac{3}{4} 2^r \cdot 2^3 \left[\frac{\cancel{r}+3-\cancel{r}-2}{(r+2)(r+3)}\right]$$

$$= \frac{3}{\cancel{4}} 2^r \cdot \cancel{8}^{2}\left[\frac{1}{(r+2)(r+3)}\right]$$

$$E[x^r] = \frac{6 \cdot 2^r}{(r+2)(r+3)} \quad \text{—— (1)}$$

To find mean $E[x]$ & $var(x)$

put $r=1$ in (1) $E[x] = \frac{6 \cdot 2}{3 \cdot 4}$
$= 1$

Put $r=2$ in (1) $E[x^2] = \frac{6 \cdot 2^2}{4 \times 5}$
$= 6/5$

$\therefore$ Mean $E[x] = 1$

& $var(x) = E[x^2] - [E[x]]^2$
$= 6/5 - 1$
$= \frac{6-5}{5} = 1/5$ //

Moment Generating function [MGF]

Moment generating function of a random Variable x about the origin is defined as

$$M_X(t) = E[e^{tx}] = \sum e^{tx} p(x), \text{ if } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx, \text{ if } x \text{ is continuous}.$$

Ily MGF about mean μ is defined by

$$M_{X-\mu}(t) = E[e^{t(x-\mu)}] = \sum e^{t(x-\mu)} p(x), \text{ if } x \text{ is discrete}$$

$$= \int e^{t(x-\mu)} f(x)\,dx \text{ if } x \text{ is continuous}$$

1. Prove that the r^{th} moment of the R.V 'x' about origin $M_X(t) = \sum \frac{t^r}{r!} \mu_r'$.

Proof:

we know that

$$M_X(t) = E[e^{tx}]$$

$$= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots \frac{(tx)^r}{r!} + \cdots\right]$$

$$\left[\text{Formula } e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots\right]$$

$$= E[1] + \frac{E[tx]}{1!} + E\left[\frac{t^2x^2}{2!}\right] + \cdots E\left[\frac{t^r x^r}{r!}\right] + \cdots$$

$$= 1 + t\,E[x] + \frac{t^2}{2!} E[x^2] + \cdots \frac{t^r}{r!} E[x^r] + \cdots$$

$$M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots \frac{t^r}{r!}\mu_r' + \cdots$$

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

Note: Thus r^{th} moment = coefficient of $\frac{t^r}{r!}$

Note: 2 $\mu_r' = \frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0}$.

Properties of MGF:

$M_X(t) = E[e^{tx}]$
$M_{X-a}(t) = E[e^{t(x-a)}]$

1. $M_{X-a}(t) = e^{-at} M_X(t)$

Proof: $M_{X-a}(t) = E[e^{t(x-a)}]$
$= e^{-at}E[e^{tx}]$
$= e^{-at}M_X(t)$

2. Let $Y = aX+b$, where X is a R.V with moment generating fun. $M_X(t)$. Then $M_Y(t) = e^{bt}M_X(at)$

Proof:
$M_Y(t) = E[e^{tY}]$
$= E[e^{t(aX+b)}]$
$= e^{bt}E[e^{X(at)}]$
$= e^{bt}M_X(at)$

3. If $M_X(t) = E[e^{tX}]$ then $M_{cX}(t) = M_X(ct)$

Soln: $M_{cX}(t) = E[e^{t(cX)}] = E[e^{(ct)X}] = M_X(ct)$

4. If X & Y are two independent RV, then $M_{X+Y}(t) = M_X(t).M_Y(t)$

Proof: $M_{X+Y}(t) = E[e^{t(X+Y)}]$
$= E[e^{tX+tY}]$
$= E[e^{tX}.e^{tY}]$
$= E[e^{tX}].E[e^{tY}]$ [$\because$ X & Y are independent]

$E[XY] = E[X].E[Y]$

X & Y

$M_{X+Y}(t) = M_X(t).M_Y(t)$

Note: If $X_1, X_2, \ldots X_n$ are n independent RVs then
$M_{X_1+X_2+\cdots+X_n}(t) = M_{X_1}(t).M_{X_2}(t)\ldots M_{X_n}(t)$.

(21)

Problems based on MGF [Discrete case]

1\. A random variable x has probability function $p(x) = \frac{1}{2^x}$; $x = 1, 2, 3 \dots$. Find the MGF, Mean & Variance.

Soln:

W.k.T $M_x(t) = \sum e^{tx} p(x)$ (for discrete R.V) $\boxed{M_x(t) = E[e^{tx}]}$

$$= \sum e^{tx} \frac{1}{2^x}$$

$$= \sum \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots$$

$$= \frac{e^t}{2}\left\{1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \dots\right\}$$

$$\therefore M_x(t) = \frac{e^t}{2} \cdot \frac{1}{1 - e^t/2} = \frac{e^t}{2 - e^t}$$

$$\mu_1' = M_x'(0) = \frac{d}{dt}\left[\frac{e^t}{2 - e^t}\right]_{t=0}$$

$$= \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t - e^{2t} + e^{2t}}{(2 - e^t)^2}\right]_{t=0}$$

$$\mu_1' = 2$$

$$\mu_2' = \frac{d^2}{dt^2}[M_x(t)]_{t=0}$$

$$= \frac{(2 - e^t)^2(2e^t) - 4e^t(2 - e^t)(-e^t)}{(2 - e^t)^4}$$

$$= \left[\frac{(2-e^t)2e^t + 4e^{2t}}{(2-e^t)^3}\right]_{t=0}$$

$$= 2+4 = 6$$

$\therefore$ Var $= \mu_2' - (\mu_1')^2$

$= 6 - 4 = 2$

$\therefore$ Mean $= 2$ & Variance $= 2$

2. Find the MGF for the distribution with

$$p(x) = \begin{cases} 2/3, & \text{at } x=1 \\ 1/3, & \text{at } x=2 \\ 0, & \text{otherwise} \end{cases}$$

Also find mean & variance.

Sol:

X takes the values 1 & 2 with probabilities 2/3 & 1/3 respectively.

$$M_X(t) = \sum_{x=1}^{2} e^{tx} p(x)$$

$$= \frac{2}{3} e^t + \frac{1}{3} e^{2t}$$

$$\therefore M_X(t) = \frac{2}{3} e^t + \frac{1}{3} e^{2t}$$

$$M_X'(t) = \frac{2}{3} e^t + \frac{2}{3} e^{2t} \quad \text{--- (1)}$$

$$M_X''(t) = \frac{2}{3} e^t + \frac{4}{3} e^{2t} \quad \text{--- (2)}$$

(1) $\Rightarrow \mu_1' = M_X'(t)\big|_{t=0} = 4/3$

(2) $\Rightarrow \mu_2' = M_X''(t)\big|_{t=0} = 6/3 = 2$

Hence mean $\mu_1' = 4/3$ &

Variance $= \mu_2' - \mu_1'^2$

$= 2 - 16/9 = 2/9$

Problems based on MGF [continuous case]

1. Find the MGF of a random variable having the pdf $f(x) = \begin{cases} 1/3, & -1<x<2 \\ 0, & \text{otherwise.} \end{cases}$

Soln:

$$M_X(t) = E[e^{tx}] = \int e^{tx} f(x)\,dx$$

$$= \int_{-1}^{2} e^{tx} \frac{1}{3}\,dx = \frac{1}{3}\left[\frac{e^{tx}}{t}\right]_{-1}^{2}$$

$$= \frac{1}{3t}\left[e^{2t} - e^{-t}\right]_{t \neq 0}$$

For $t=0$ $M_X(0) = \lim_{t\to 0} \frac{e^{2t}-e^{-t}}{3t} = \frac{0}{0}$

$$= \lim_{t\to 0} \frac{2e^{2t}+e^{-t}}{3} = 3/3 = 1$$

$$\therefore M_X(t) = \begin{cases} \frac{e^{2t}-e^{-t}}{3t}, & t \neq 0 \\ 1, & t=0 \end{cases}$$

2. Find the MGF of a r.v X whose pdf is defined by $f(x) = \begin{cases} x, & \text{for } 0 \le x \le 1 \\ 2-x, & \text{for } 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$ hence find mean & var. of X

Soln:

$$M_X(t) = \int e^{tx} f(x)\,dx$$

$$= \int_0^1 x e^{tx}\,dx + \int_1^2 (2-x) e^{tx}\,dx$$

$$= \left[x\left[\frac{e^{tx}}{t}\right] - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} + \frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left[\frac{e^{t}}{t} - \frac{e^{t}}{t^2} - 0 + \frac{1}{t^2}\right] + \left[0 + \frac{e^{2t}}{t^2} - \frac{e^{t}}{t} - \frac{e^{t}}{t^2}\right]$$

$$= \left[\frac{1-2e^{t}+e^{2t}}{t^2}\right] = \left(\frac{1-e^{t}}{t}\right)^2$$

To find mean & variance.

$$M_X(t) = \left[\frac{1-e^t}{-t}\right]^2 = \left[\frac{1-e^t}{t}\right]^2 \qquad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$= \left\{\frac{1 - \left[1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots\right]}{t}\right\}^2$$

$$= \left\{\frac{-\left[\frac{t}{1} + \frac{t^2}{2} + \frac{t^3}{6} + \cdots\right]}{t}\right\}^2$$

$$M_X(t) = \left[1 + \frac{t}{2} + \frac{t^2}{6} + \cdots\right]^2$$

$$\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = M_X'(0)$$

$$M_X'(t) = 2\left(1 + \frac{t}{2} + \frac{t^2}{6} + \cdots\right)\left[\frac{1}{2} + \frac{t}{3} + \cdots\right] \quad \text{--- (1)}$$

$$M_X'(0) = 2(1)(1/2) = 1 \Rightarrow \boxed{\text{Mean} = 1}$$

$$(1) \Rightarrow M_X''(t) = 2\left(1 + \frac{t}{2} + \frac{t^2}{6} \cdots\right)(1/3 + \cdots) + 2\left(\frac{1}{2} + \frac{t}{3} + \cdots\right)(1/2 + t/3 + \cdots)$$

$$M_X''(0) = 2[1][1/3] + 2(1/2)(1/2)$$

$$= 2/3 + 1/2 = \frac{4+3}{6} = 7/6$$

$$\text{Variance of } X = E[X^2] - [E(X)]^2$$

$$= \mu_2' - \mu_1^2$$

$$= 7/6 - 1 = 1/6$$

Hence $\boxed{\text{Mean} = 1}$ & $\boxed{\text{Variance} = 1/6}$