Topic:5.6 – EULERS(CAUCHYS) LINEAR EQUATIONS

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a_n y = f(x) \rightarrow (1)$$

where $a_1, a_2, \cdots a_n$ are constants and $f(x)$ is a function of x.

(1) can be reduced to linear differential equation with constant coefficient by putting the sub.

$x = e^z$ (or) $z = \log x$

$$x \frac{dy}{dx} = D'y \quad \text{where} \quad D' = \frac{d}{dz}$$

$$x^2 \frac{d^2y}{dx^2} = D'(D'-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y$$

1) Solve $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 0$

Given $(x^2 D^2 + xD)\, y = 0$

Put $x = e^z \qquad \log x = z$

$xD = D' \qquad x^2 D^2 = D'(D'-1)$

$[D'(D'-1) + D']\, y = 0$

$[D'^2 - D' + D']\, y = 0 \Rightarrow D'^2 y = 0$

AE $m^2 = 0$; $m = 0, 0$

$CF = [A + Bz]\, e^{0z}$

$= A + Bz$

$CF = A + B \log x$

$y = A + B \log x \quad [\because RHS = 0 \quad PI = 0]$

2) Solve: $(x^2D^2 - 2xD - 4)y = x^2 + 2\log x$

Put $x = e^z \quad \log x = z$

$xD = D' \quad x^2D^2 = D'(D'-1)$

$[D'(D'-1) - 2D' - 4]y = e^{2z} + 2z$

$[D'^2 - D' - 2D' - 4]y = e^{2z} + 2z$

$[D'^2 - 3D' - 4]y = e^{2z} + 2z$

AE $m^2 - 3m - 4 = 0 \Rightarrow (m-4)(m+1) = 0$

$m = -1, 4$

$CF = Ae^{-z} + Be^{4z}$

$= Ax^{-1} + Bx^4 = \frac{A}{x} + Bx^4$

$PI_1 = \frac{1}{D'^2 - 3D' - 4} e^{2z} = \frac{1}{4-6-4} e^{2z}$

$= \frac{1}{-6} e^{2z} = -\frac{1}{6} x^2$

$PI_2 = \frac{1}{D'^2 - 3D' - 4} 2z = 2 \frac{1}{-4\left(1 + \frac{3D' - D'^2}{4}\right)} z$

$$= \frac{1}{-6} e^{2z} = -\frac{1}{6} x^2$$

$$PI_2 = \frac{1}{D'^2 - 3D' - 4} 2z = 2 \frac{1}{-4\left(1 + \frac{3D' - D'^2}{4}\right)} z$$

$$= -\frac{1}{2} \left[1 + \frac{3D' - D'^2}{4}\right]^{-1} z$$

$$= -\frac{1}{2} \left[1 - \left(\frac{3D' - D'^2}{4}\right) + \cdots\right] z$$

$$= -\frac{1}{2} \left[1 - \frac{3D'}{4} + \frac{D'^2}{4}\right] z$$

$$= -\frac{1}{2} \left[z - \frac{3}{4}\right] = -\frac{z}{2} + \frac{3}{8}$$

$$PI_2 = -\frac{1}{2} \log x + \frac{3}{8}$$

$$\therefore y = \frac{A}{x} + Bx^4 - \frac{1}{6}x^2 - \frac{1}{2}\log x + \frac{3}{8}$$