Topic: 3. 5 – EVOLUTES

5.

Involutes and Evolutes.

Involutes and Evolutes:

The locus of the centre of curvature of the given curve is called the evolute of the curve. The given curve is called the involute of the evolute.

Working rule to find Evolute:

1. Write the parametric equation of the given curve.
2. Find the centre of curvature $= (\bar{x}, \bar{y})$.
3. Eliminate θ the parameter θ (or) t from $(\bar{x}, \bar{y})$
4. taking the locus of $(\bar{x}, \bar{y})$ the required evolute is $g(x, y) = c$.

curve	cartesian equation	parametric equation.
parabola	1. $y^2 = 4ax$ 2. $x^2 = 4ay$	1. $x = at^2;\ y = 2at$ 2. $x = 2at;\ y = at^2$.
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$x = a\cos\theta,$ $y = b\sin\theta$
hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$x = a\sec\theta;\ y = b\tan\theta$.
Rectangular hyperbola	$xy = c^2$	$x = ct,\ y = c/t$.
Astroid	$x^{2/3} + y^{2/3} = a^{2/3}$	$x = a\cos^3\theta,\ y = a\sin^3\theta$.

1. Find the equation of the evolute of the parabola $y^2 = 4ax$.

Soln:

The parametric equation of parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$.

We have to find the centre of curvature

$x = at^2$; $y = 2at$.

$\frac{dx}{dt} = 2at$; $\frac{dy}{dt} = 2a$.

$y_1 = \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2a}{2at} = \frac{1}{t}$.

$y_2 = \frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right)\frac{dt}{dx}$

$= -\frac{1}{t^2} \cdot \frac{1}{2at} = -\frac{1}{2at^3}$.

$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$.

$= at^2 - \frac{1}{t}\left(\frac{-2at^3}{1}\right) \cdot \left(1 + \frac{1}{t^2}\right)$

$= at^2 + 2at^3\left(\frac{1}{t}\right)\left(\frac{t^2+1}{t^2}\right)$

$= at^2 + 2a(t^2+1) = at^2 + 2at^2 + 2a$

$\bar{x} = 3at^2 + 2a \rightarrow (1)$.

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 2at + (-2at^3)\left(1+\frac{1}{t^2}\right)$$

$$= 2at - 2at^3\left(\frac{t^2+1}{t^2}\right) = 2at - 2at^3\frac{(t^2+1)}{t^2}.$$

$$= 2at - 2at(t^2+1)$$

$$\bar{y} = 2at - 2at^3 - 2at$$

$$\bar{y} = -2at^3 \longrightarrow (2)$$

Now we have to eliminate 't' between (1) & (2)

$$(1) \Rightarrow t^2 = \frac{\bar{x} - 2a}{3a} \rightarrow (3)$$

$$t^6 = \left(\frac{\bar{x} - 2a}{3a}\right)^3$$

$$t^6 = \frac{(\bar{x} - 2a)^3}{27a^3} \rightarrow (5)$$

$$(2) \Rightarrow t^3 = \frac{\bar{y}}{-2a} \rightarrow (4)$$

Squaring (4) we get

$$t^6 = \left(\frac{\bar{y}}{-2a}\right)^2$$

$$t^6 = \frac{\bar{y}^2}{4a^2} \longrightarrow (6)$$

From (5) and (6).

$$\frac{\bar{y}^2}{4a^2} = \frac{(\bar{x} - 2a)^3}{27a^3}$$

$$\frac{\bar{y}^2}{4} = \frac{(\bar{x} - 2a)^3}{27a}$$

$$27a\bar{y}^2 = 4(\bar{x} - 2a)^3$$

changing $\bar{x}$ and $\bar{y}$ to x and y the locus of $(\bar{x}, \bar{y})$ becomes $27ay^2 = 4(x-2a)^3$, which gives the evolute of the parabola $y^2 = 4ax$.

2. find the equation of the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln: The parametric equations of the ellipse are $x = a\cos\theta$; $y = b\sin\theta$.

$\frac{d}{d\theta}(\cot\theta) = -\operatorname{cosec}^2\theta$

$$\frac{dx}{d\theta} = -a\sin\theta; \quad \frac{dy}{d\theta} = b\cos\theta$$

$$y_1 = -\frac{b}{a}\cot\theta; \quad y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\frac{d\theta}{dx}$$

$$= \frac{b}{a}\operatorname{cosec}^2\theta\left(\frac{-1}{a\sin\theta}\right)$$

$$y_2 = -\frac{b}{a^2}\operatorname{cosec}^3\theta.$$

Let $(\bar{x}, \bar{y})$ be the centre of curvature.

$$\bar{x} = x - \frac{y_1}{y_2}(1 + y_1^2)$$

$$= a\cos\theta - \left[\frac{-b}{a}\cot\theta\right]\left[\frac{-a^2}{b}\sin^3\theta\right]\left[1 + \frac{b^2}{a^2}\cot^2\theta\right]$$

$$= a\cos\theta - \frac{a\cos\theta}{\sin\theta}\sin^3\theta\left[1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]$$

$$= a\cos\theta - a\cos\theta\sin^2\theta\frac{(a^2\sin^2\theta + b^2\cos^2\theta)}{a^2\sin^2\theta}$$

$$= a\cos\theta - \frac{\cos\theta}{a}(a^2\sin^2\theta + b^2\cos^2\theta)$$

(6)

$$= a\cos\theta - a\sin^2\theta\cos\theta - \frac{b^2}{a}\cos^3\theta.$$

$$= a\cos\theta - a(1-\cos^2\theta)\cos\theta - \frac{b^2}{a}\cos^3\theta.$$

$$= a\cos\theta - a\cos\theta + a\cos^3\theta - \frac{b^2}{a}\cos^3\theta.$$

$$\bar{x} = \left(\frac{a^2-b^2}{a}\right)\cos^3\theta. \rightarrow (1)$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = b\sin\theta - \frac{a^2}{b}\sin^3\theta\left(1+\frac{b^2}{a^2}\cot^2\theta\right)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= b\sin\theta - \frac{\sin\theta}{b}(a^2\sin^2\theta + b^2\cos^2\theta)$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\cos^2\theta\sin\theta$$

$$= b\sin\theta(1-\cos^2\theta) - \frac{a^2}{b}\sin^3\theta$$

$$= b\sin\theta\sin^2\theta - \frac{a^2}{b}\sin^3\theta$$

$$\bar{y} = \left(\frac{b^2-a^2}{b}\right)\sin^3\theta \rightarrow (2)$$

Now we have to eliminate θ between (1) + (2)

$$(1) \Rightarrow ax = (a^2-b^2)\cos^3\theta$$

$$(ax)^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta \rightarrow (3).$$

$$(2) \Rightarrow by = (b^2-a^2)\sin^3\theta$$

$$(by)^{2/3} = (b^2-a^2)^{2/3}\sin^2\theta \rightarrow (4)$$

(3) + (4) $\Rightarrow (ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}\left[\sin^2\theta + \cos^2\theta\right]$

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

change $\bar{x}$ & $\bar{y}$ to x and y the locus $q(\bar{x}, \bar{y})$ becomes $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ which gives the evolute of the ellipse.