Topic:4.9–LAGRANGE'S METHOD

Using Lagrange's method of multipliers, show that the stationary value of $a^3x^2+b^3y^2+c^3z^2$ where $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$ occur at $x=\frac{a+b+c}{a}$; $y=\frac{a+b+c}{b}$; $z=\frac{a+b+c}{c}$.

Solution: Let the Auxiliary function F be

$$F(x,y,z,\lambda)=(a^3x^2+b^3y^2+c^3z^2)+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\right.$$

Where λ is Lagrange multiplier.

$F_x=\frac{\partial F}{\partial x}$ $\quad$ $F_y=\frac{\partial F}{\partial y}$ $\quad$ $F_z=\frac{\partial F}{\partial z}$

$=2a^3x-\frac{\lambda}{x^2}$ $\quad$ $=2b^3y-\frac{\lambda}{y^2}$ $\quad$ $=2c^3z-\frac{\lambda}{z^2}$

To find the stationary value.

$F_x = 0$

$2a^3x - \frac{\lambda}{x^2} = 0$

$2a^3x = \frac{\lambda}{x^2}$

$a^3x^3 = \frac{\lambda}{2}$

$ax = \left(\frac{\lambda}{2}\right)^{1/3}$ → (1)

$F_y = 0$

$2b^3y - \frac{\lambda}{y^2} = 0$

$2b^3y = \frac{\lambda}{y^2}$

$b^3y^3 = \frac{\lambda}{2}$

$by = \left(\frac{\lambda}{2}\right)^{1/3}$ → (2)

$F_z = 0$.

$2c^3z - \frac{\lambda}{z^2} = 0$.

$2c^3z = \frac{\lambda}{z^2}$

$c^3z^3 = \frac{\lambda}{2}$

$cz = \left(\frac{\lambda}{2}\right)^{1/3}$. → (3).

from (1), (2), (3), we get.

$ax = by = cz$.

Given $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

$\frac{a}{ax} + \frac{b}{by} + \frac{c}{cz} = 1$ [since $ax = by = cz$]

$\frac{a+b+c}{ax} = 1 \Rightarrow x = \frac{a+b+c}{a}$.

$y = \frac{a+b+c}{b}$; $z = \frac{a+b+c}{c}$.

∴ f is stationary at this point.

3) Find the point on the plane $ax+by+cz=p$ at which $f=x^2+y^2+z^2$ has a stationary value and find the stationary values of f, using Lagrange's method of multipliers.

Sol: Let the auxiliary function F be

$$F(x,y,z,\lambda)=x^2+y^2+z^2+\lambda(ax+by+cz-p)$$

where λ is Lagrange multiplier.

$$F_x=\frac{\partial F}{\partial x}=2x+\lambda a \qquad F_y=\frac{\partial F}{\partial y}=2y+\lambda b \qquad F_z=\frac{\partial F}{\partial z}=2z+\lambda c.$$

To find the stationary values,

$$F_x=0 \qquad F_y=0 \qquad F_z=0$$

$$2x+\lambda a=0 \qquad 2y+\lambda b=0 \qquad 2z+\lambda c=0$$

$$2x=-\lambda a \qquad 2y=-\lambda b \qquad 2z=-\lambda c$$

$\frac{x}{a} = -\frac{\lambda}{2} \rightarrow (1)$ $\quad \frac{y}{b} = -\frac{\lambda}{2} \rightarrow (2)$ $\quad \frac{z}{c} = -\frac{\lambda}{2} \rightarrow (3)$

from (1), (2), (3) we get:

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

$$\frac{ax}{a^2} = \frac{by}{b^2} = \frac{cz}{c^2},$$

$$\Rightarrow \frac{ax}{a^2} = \frac{by}{b^2} = \frac{cz}{c^2} = \frac{ax+by+cz}{a^2+b^2+c^2} = \frac{P}{a^2+b^2+c^2}$$

$$x = \frac{aP}{a^2+b^2+c^2}\ ;\quad y = \frac{bP}{a^2+b^2+c^2}\ ;\quad z = \frac{cP}{a^2+b^2+c^2},$$

Stationary value of $f = x^2 + y^2 + z^2$.

$$= \left(\frac{ap}{a^2+b^2+c^2}\right)^2 + \left(\frac{bp}{a^2+b^2+c^2}\right)^2 + \left(\frac{cp}{a^2+b^2+c^2}\right)^2$$

$$= \frac{a^2p^2 + b^2p^2 + c^2p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{(a^2+b^2+c^2)\,p^2}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2}{a^2+b^2+c^2}$$

We can prove this to be minimum.