Topic: 3. 6 – EVOLUTES

6. Find the evolute of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Soln:

The parametric equation of the hyperbola.

$x = a\sec\theta$; $y = b\tan\theta$.

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$; $\frac{dy}{d\theta} = b\sec^2\theta$.

$y_1 = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a} \cdot \frac{1}{\cos\theta} \cdot \frac{\cos\theta}{\sin\theta}$

$y_1 = \frac{b}{a}\operatorname{cosec}\theta$

$y_2 = \frac{d}{d\theta}\left(\frac{b}{a}\operatorname{cosec}\theta\right) \cdot \frac{d\theta}{dx}$

$\frac{d}{d\theta}\operatorname{cosec}\theta = -\operatorname{cosec}\theta\cot\theta$

$= -\frac{b}{a}\operatorname{cosec}\theta \cdot \cot\theta \cdot \frac{1}{a\sec\theta\tan\theta}$

$y_2 = -\frac{b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}$

$$\bar{x} = x - \frac{y_1}{y_2}(1+y_1^2)$$

$$= a\sec\theta - \frac{b}{a}\operatorname{cosec}\theta \left(\frac{-a^2}{b}\right)\frac{\sin^3\theta}{\cos^3\theta}\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)$$

$$= a\sec\theta + a\frac{1}{\sin\theta}\frac{\sin^3\theta}{\cos^3\theta}\left[a^2+\frac{b^2}{a^2}\frac{1}{\sin^2\theta}\right]$$

$$= a\sec\theta + a\frac{\sin^2\theta}{\cos^3\theta}\left[\frac{a^2\sin^2\theta+b^2}{a^2\sin^2\theta}\right]$$

$$= a\sec\theta + \frac{1}{a\cos^3\theta}\left[a^2(1-\cos^2\theta)+b^2\right]$$

$$= a\sec\theta + \frac{1}{a\cos^3\theta}\left[a^2-a^2\cos^2\theta+b^2\right]$$

$$\bar{x} = a\sec\theta + \frac{1}{a}\sec^3\theta\left[a^2-a^2\cos^2\theta+b^2\right]$$

$$a\bar{x} = a^2\sec\theta + \sec^3\theta\left[a^2-a^2\cos^2\theta+b^2\right]$$

$$= a^2\sec\theta + a^2\sec^3\theta - a^2\cos^2\theta\sec^3\theta + b^2\sec^3\theta$$

$$= a^2\sec\theta + a^2\sec^3\theta - a^2\sec\theta + b^2\sec^3\theta.$$

$$a\bar{x} = (a^2+b^2)\sec^3\theta.$$

$$(a\bar{x})^{2/3} = (a^2+b^2)^{2/3}\sec^2\theta.$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= b\tan\theta + \left(-\frac{a^2}{b}\frac{\sin^3\theta}{\cos^3\theta}\right)\left(1+\frac{b^2}{a^2}\operatorname{cosec}^2\theta\right)$$

$$= b\tan\theta - \frac{a^2}{b}\tan^3\theta\,\frac{\left[a^2+b^2\operatorname{cosec}^2\theta\right]}{a^2}.$$

$$\bar{y} = b\tan\theta - \frac{\tan^3\theta}{b}\left[a^2+b^2\operatorname{cosec}^2\theta\right]$$

$$b\bar{y} = b^2\tan\theta - a^2\tan^3\theta - b^2\tan^3\theta\operatorname{cosec}^2\theta$$

$$= b^2\tan\theta - a^2\tan^3\theta - b^2\tan\theta\sec^2\theta$$

$$= b^2\tan\theta - a^2\tan^3\theta - b^2\tan\theta(1+\tan^2\theta)$$

$$b\bar{y} = -(a^2+b^2)\tan^3\theta$$

$$(b\bar{y})^{2/3} = (a^2+b^2)^{2/3}\tan^2\theta.$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}(\sec^2\theta - \tan^2\theta)$$

$$(a\bar{x})^{2/3} - (b\bar{y})^{2/3} = (a^2+b^2)^{2/3}.$$

changing $\bar{x}$ and $\bar{y}$ by x and y

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$$