



TOPIC: 4.1 – PARTIAL DERIVATIVE AND TOTAL DERIVATIVES

Problems based on Partial derivatives

1. If $u = (x-y)(y-z)(z-x)$, then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Given $u = (x-y)(y-z)(z-x)$

$$\frac{\partial u}{\partial x} = (y-z) [(x-y)(1) + (z-x)(1)]$$

$$= -(x-y)(y-z) + (y-z)(z-x)$$

$$\frac{\partial u}{\partial y} = (z-x) [(x-y)(1) + (y-z)(-1)]$$

$$\frac{\partial u}{\partial y} = (z-x) [(x-y)(1) + (y-z)(-1)]$$

$$= (x-y)(z-x) - (y-z)(z-x)$$

$$\frac{\partial u}{\partial z} = (x-y) [(y-z)(+1) + (z-x)(-1)]$$

$$= (x-y)(y-z) - (x-y)(z-x)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$



Euler's Theorem for homogeneous function

If u is a homogeneous function of degree n in x and y , then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

① If $u = \sin^{-1} \left[\frac{x^3 - y^3}{x+y} \right]$, then prove that
 $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u$.

$$\text{Let } f(x, y) = \sin u = \frac{x^3 - y^3}{x+y}$$

$$f(tx, ty) = \frac{t^3 x^3 - t^3 y^3}{tx+ty} = t^2 f(x, y)$$

$\therefore f$ is a homogeneous function of degree 2 in x & y .

\therefore By Euler's theorem, we get

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf = 2f \rightarrow \text{①}$$



AN AUTONOMOUS INSTITUTION

Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A' Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

Here $f = \sin u$

sub. in (1),

$$x \frac{\partial}{\partial x} (\sin u) + y \frac{\partial}{\partial y} (\sin u) = 2 \sin u$$

$$x \left[\cos u \frac{\partial u}{\partial x} \right] + y \left[\cos u \frac{\partial u}{\partial y} \right] = 2 \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u$$

(2) If $u = \frac{x}{y} + \frac{y}{x} + \frac{z}{x}$, then find

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$$

By Euler's Theorem

$$\text{Given } u(x, y, z) = \frac{x}{y} + \frac{y}{x} + \frac{z}{x}$$

$$u(tx, ty, tz) = \frac{tx}{ty} + \frac{ty}{tx} + \frac{tz}{tx}$$

$$= t^0 u(x, y, z)$$

$\therefore u$ is a homogeneous function of x, y, z
in degree 0.

\therefore By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu = 0$$



④ If $u = \cos^{-1} \left[\frac{x+y}{\sqrt{x}+\sqrt{y}} \right]$, then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \cot u.$$

$$\text{Let } f(x, y) = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

$$f(tx, ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = t^{\frac{1}{2}} f(x, y)$$

$\Rightarrow f$ is a homogeneous function of degree $\frac{1}{2}$ in x and y .



∴ By Euler's theorem,

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf = \frac{1}{2} f \rightarrow \textcircled{1}$$

Here $f = \cos u$, sub in $\textcircled{1}$,

$$x \frac{\partial}{\partial x} (\cos u) + y \frac{\partial}{\partial y} (\cos u) = \frac{1}{2} \cos u$$

$$x \left[-\sin u \frac{\partial u}{\partial x} \right] + y \left[-\sin u \frac{\partial u}{\partial y} \right] = \frac{1}{2} \cos u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \frac{\cos u}{\sin u}$$

$$= -\frac{1}{2} \cot u$$



AN AUTONOMOUS INSTITUTION

Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A' Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

① If $u = (x^2 + y^2 + z^2)^{-1/2}$, then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Given $u = (x^2 + y^2 + z^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2} (x^2 + y^2 + z^2)^{-3/2} (2x)$$
$$\frac{\partial u}{\partial x} = -x (x^2 + y^2 + z^2)^{-3/2}$$
$$\frac{\partial^2 u}{\partial x^2} = -\left[x \left(-\frac{3}{2}\right) (x^2 + y^2 + z^2)^{-5/2} (2x) + (x^2 + y^2 + z^2)^{-3/2} \right]$$
$$\frac{\partial^2 u}{\partial x^2} = 3x^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} \rightarrow \textcircled{1}$$

ii) $\frac{\partial^2 u}{\partial y^2} = 3y^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} \rightarrow \textcircled{2}$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2 (x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} \rightarrow \textcircled{3}$$

① + ② + ③

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = (x^2 + y^2 + z^2)^{-5/2} [3x^2 + 3y^2 + 3z^2] - 3(x^2 + y^2 + z^2)^{-3/2}$$



Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A' Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

$$\begin{aligned} &= 3(x^2 + y^2 + z^2)(x^2 + y^2 + z^2)^{-5/2} - 3(x^2 + y^2 + z^2)^{-3/2} \\ &= 3(x^2 + y^2 + z^2)^{-3/2} - 3(x^2 + y^2 + z^2)^{-3/2} \\ &= 0 \end{aligned}$$

Total Derivatives

If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$
Then we can express u as a function of t
alone by substituting the values of x and y in
 $f(x, y)$. Thus, we can find the ordinary
derivative $\frac{du}{dt}$ which is called the total derivative
of u to distinguish it from the partial derivatives
 $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. i) $\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$



Composite function of one variable

If $u = f(x, y, z)$ where x, y, z are all functions of a variable t , then

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$$

Differentiation of Implicit functions

If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x , then

$$\frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \quad \left[\because \frac{\partial f}{\partial y} \neq 0 \right]$$

Composite function of two variables

If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$, then z is a function of u, v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$$



AN AUTONOMOUS INSTITUTION

Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A' Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

① Find $\frac{dy}{dx}$ when $x^3 + y^3 = 3axy$.

$$\text{Let } f(x, y) = x^3 + y^3 - 3axy$$

$$\frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = - \frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$= - \frac{x^2 - ay}{y^2 - ax}$$

⑤ If $Z = f(y-z, z-x, x-y)$, show that

$$\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = 0$$

$$\text{Let } u = y - z, \quad v = z - x, \quad w = x - y$$

$$Z = f(u, v, w)$$

$$\frac{\partial Z}{\partial x} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u} (0) + \frac{\partial f}{\partial v} (-1) + \frac{\partial f}{\partial w} (1)$$

$$= - \frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

$$\text{Similarly } \frac{\partial Z}{\partial y} = + \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$$



11) If $u = \log(x^2 + y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, prove that

$$u_{xx} + u_{yy} = 0.$$

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} (2x) + \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left[-\frac{y}{x^2} \right]$$

$$= \frac{2x}{x^2 + y^2} + \frac{1}{\frac{x^2 + y^2}{x^2}} \left[-\frac{y}{x^2} \right]$$

$$= \frac{2x}{x^2 + y^2} - \frac{y}{x^2 + y^2} = \frac{2x - y}{x^2 + y^2}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2)(2) - (2x - y)(2x)}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4x^2 + 2xy}{(x^2 + y^2)^2}$$



AN AUTONOMOUS INSTITUTION

Accredited by NBA – AICTE and Accredited by NAAC – UGC with 'A' Grade
Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

$$u_{xx} = \frac{2y^2 - 2x^2 + 2xy}{(x^2 + y^2)^2} \rightarrow \textcircled{1}$$

$$\begin{aligned} u_y &= \frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} (2y) + \frac{1}{1 + (\frac{y}{x})^2} \left(\frac{1}{x}\right) \\ &= \frac{2y}{x^2 + y^2} + \frac{1}{\frac{x^2 + y^2}{x^2}} \left(\frac{1}{x}\right) \\ &= \frac{2y}{x^2 + y^2} + \frac{x}{x^2 + y^2} = \frac{2y + x}{x^2 + y^2} \end{aligned}$$

$$u_{yy} = \frac{(x^2 + y^2)(2) - (2y + x)(2y)}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4y^2 - 2xy}{(x^2 + y^2)^2}$$

$$u_{yy} = \frac{2x^2 - 2y^2 - 2xy}{(x^2 + y^2)^2} \rightarrow \textcircled{2}$$

$$\textcircled{1} + \textcircled{2}$$

$$u_{xx} + u_{yy} = \frac{2y^2 - 2x^2 + 2xy + 2x^2 - 2y^2 - 2xy}{(x^2 + y^2)^2}$$

$$= 0$$