



AN AUTONOMOUS INSTITUTION

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**TOPIC: 4.2 - DIFFERENTIATION OF IMPLICIT FUNCTIONS,
JACOBIAN AND PROPERTIES**

Jacobians

If u_1, u_2, \dots, u_n are functions of n variables x_1, x_2, \dots, x_n , then the Jacobian of a transformation from x_1, x_2, \dots, x_n to u_1, u_2, \dots, u_n is defined by

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \dots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \dots & \frac{\partial u_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \dots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

and is denoted by the symbol $\frac{\partial(u_1, u_2, \dots, u_n)}{\partial(x_1, x_2, \dots, x_n)}$
or $J_{(u_1, u_2, \dots, u_n)}$

In particular $\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix}$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$



Functional dependence

If u, v, w are functionally dependent functions of the independent variables x, y, z

then
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

① If $x = r \cos \theta$, $y = r \sin \theta$, find (i) $\frac{\partial(x, y)}{\partial(r, \theta)}$

(ii) $\frac{\partial(r, \theta)}{\partial(x, y)}$

Given $x = r \cos \theta$

$$y = r \sin \theta$$

$$\frac{\partial x}{\partial r} = \cos \theta$$

$$\frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta$$



$$\begin{aligned} \text{(i)} \quad \frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r \cos^2 \theta + r \sin^2 \theta = r \end{aligned}$$

$$\text{(ii)} \quad \frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}$$



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⑥ Find the Jacobian of y_1, y_2, y_3 w.r.t x_1, x_2, x_3 if $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= -\frac{x_2 x_3}{x_1^2} \left[\frac{x_1^2 x_2 x_3}{x_2^2 x_3^2} - \frac{x_1^2}{x_2 x_3} \right] - \frac{x_3}{x_1} \left[-\frac{x_1 x_2 x_3}{x_2 x_3^2} - \frac{x_1}{x_2} \right]$$
$$+ \frac{x_2}{x_1} \left[\frac{x_1 x_3}{x_2 x_3} + \frac{x_1 x_2 x_3}{x_2^2 x_3} \right]$$
$$= -1 + 1 + 1 + 1 + 1 + 1$$
$$= 4$$