



TOPIC: 4.3 – PROBLEMS ON JACOBIAN

(2) Find the Jacobian $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)}$ of the transformation $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$
 $z = r \cos \theta$.

Given $x = r \sin \theta \cos \phi$

$$\frac{\partial x}{\partial r} = \sin \theta \cos \phi$$

$$\frac{\partial x}{\partial \theta} = r \cos \theta \cos \phi$$

$$\frac{\partial x}{\partial \phi} = -r \sin \theta \sin \phi$$

$$y = r \sin \theta \sin \phi$$

$$\frac{\partial y}{\partial r} = \sin \theta \sin \phi$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta \sin \phi$$

$$\frac{\partial y}{\partial \phi} = r \sin \theta \cos \phi$$

$$z = r \cos \theta$$

$$\frac{\partial z}{\partial r} = \cos \theta$$

$$\frac{\partial z}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial z}{\partial \phi} = 0$$



$$\begin{aligned} \text{Jacobian } J &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \omega} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \omega} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \omega} & \frac{\partial z}{\partial \phi} \end{vmatrix} \\ &= \begin{vmatrix} \sin \omega \cos \phi & r \cos \omega \cos \phi & -r \sin \omega \sin \phi \\ \sin \omega \sin \phi & r \cos \omega \sin \phi & r \sin \omega \cos \phi \\ \cos \omega & -r \sin \omega & 0 \end{vmatrix} \\ &= \cos \omega \left[r^2 \sin \omega \cos \omega \cos^2 \phi + r^2 \sin \omega \cos \omega \sin^2 \phi \right] \\ &\quad + r \sin \omega \left[r \sin^2 \omega \cos^2 \phi + r \sin^2 \omega \sin^2 \phi \right] \end{aligned}$$



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$$\begin{aligned} &= r^2 \sin \alpha \cos^2 \alpha [\cos^2 \phi + \sin^2 \phi] \\ &\quad + r^2 \sin^3 \alpha [\cos^2 \phi + \sin^2 \phi] \\ &= r^2 \sin \alpha \cos^2 \alpha + r^2 \sin^3 \alpha \\ &= r^2 \sin \alpha [\cos^2 \alpha + \sin^2 \alpha] = \underline{\underline{r^2 \sin \alpha}} \end{aligned}$$

2) If $x = uv$ and $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.

Given $x = uv$

$$\frac{\partial x}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u$$

$$y = \frac{u}{v}$$

$$\frac{\partial y}{\partial u} = \frac{1}{v}$$

$$\frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}$$



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(8) If $u = 2xy$, $v = x^2 - y^2$ and $x = r \cos \theta$,
 $y = r \sin \theta$. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without usual
substitution.

Given $u = 2xy$

$$\frac{\partial u}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = 2x$$

$$v = x^2 - y^2$$

$$\frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial v}{\partial y} = -2y$$

$$x = r \cos \theta$$

$$\frac{\partial x}{\partial r} = \cos \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$y = r \sin \theta$$

$$\frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta$$



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$$\begin{aligned} \frac{\partial(u, v)}{\partial(r, \omega)} &= \frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(r, \omega)} \\ &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \omega} \end{vmatrix} \\ &= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos \omega & -r \sin \omega \\ \sin \omega & r \cos \omega \end{vmatrix} \\ &= (-4y^2 - 4x^2) (r \cos^2 \omega + r \sin^2 \omega) \\ &= -4(x^2 + y^2) \cdot r \\ &= -4r \cdot r^2 = -4r^3 \end{aligned}$$