TOPIC: 4.3 – PROBLEMS ON JACOBIAN

(2) Find the Jacobian $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.

Given $x = r\sin\theta\cos\phi$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \qquad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi$$

$y = r\sin\theta\sin\phi$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$$

$$\frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

$z = r\cos\theta$

$$\frac{\partial z}{\partial r} = \cos\theta$$

$$\frac{\partial z}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial z}{\partial \phi} = 0$$

Jacobian $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\left[r^2\sin\theta\cos\theta\cos^2\phi + r^2\sin\theta\cos\theta\sin^2\phi\right]$$

$$+ r\sin\theta\left[r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi\right]$$

$$= r^2 \sin\omega \cos^2\omega \left[\cos^2\phi + \sin^2\phi\right]$$
$$+ r^2 \sin^3\omega \left[\cos^2\phi + \sin^2\phi\right]$$
$$= r^2 \sin\omega \cos^2\omega + r^2 \sin^3\omega$$
$$= r^2 \sin\omega \left[\cos^2\omega + \sin^2\omega\right] = r^2 \sin\omega$$

) If $x = uv$ and $y = \frac{u}{v}$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.

Given $x = uv$

$$\frac{\partial x}{\partial u} = v$$
$$\frac{\partial x}{\partial v} = u$$

$y = \frac{u}{v}$

$$\frac{\partial y}{\partial u} = \frac{1}{v}$$
$$\frac{\partial y}{\partial v} = -\frac{u}{v^2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v}$$

(8) If $u = 2xy$, $v = x^2 - y^2$ and $x = r\cos\theta$, $y = r\sin\theta$. Evaluate $\frac{\partial(u,v)}{\partial(r,\theta)}$ without usual substitution.

Given $u = 2xy$

$$\frac{\partial u}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = 2x$$

$v = x^2 - y^2$

$$\frac{\partial v}{\partial x} = 2x$$

$$\frac{\partial v}{\partial y} = -2y$$

$x = r\cos\theta$

$$\frac{\partial x}{\partial r} = \cos\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$y = r\sin\theta$

$$\frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(u,v)}{\partial(r,\omega)} = \frac{\partial(u,v)}{\partial(x,y)} \frac{\partial(x,y)}{\partial(r,\omega)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \omega} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \omega} \end{vmatrix}$$

$$= \begin{vmatrix} 2y & 2x \\ 2x & -2y \end{vmatrix} \begin{vmatrix} \cos\omega & -r\sin\omega \\ \sin\omega & r\cos\omega \end{vmatrix}$$

$$= (-4y^2 - 4x^2)(r\cos^2\omega + r\sin^2\omega)$$

$$= -4(x^2+y^2) \cdot r$$

$$= -4r \cdot r^2 = -4r^3$$