Macaulay’s method

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Macaulay's method is suitable

i) When the beam is subjected to an eccentric point load.

ii) When the beam is subjected to a number of concentrated loads.

Changes of Macaulay’s method

1.Brackets are to be integrated as a whole.

2.Constants of integration are written after the first term.

3.The section, for which BM equation is to be written, should be taken in the last part of the beam.

Method of Singularity functions

In Macaulay’s method a single equation is formed for all loading on a beam, the equation is constructed in such a way that the constant of Integration apply to all portions of the beam. This method is also called method of singularity functions.

The rules observed using Macaulay‟s method

Always take origin on the extreme of the beam.

Take left clockwise moment as negative and left counter clockwise moment as positive.

Take a section in the least segment of the beam and take moment from the left.

If the beam carries a UDL, extend it upto the extreme right and superimpose a UDL equal and opposite to that, which has been added while extending the given UDL.

Uses of Macaulay's Method

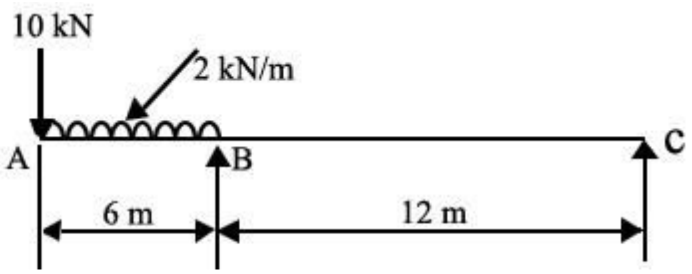
When the problem of deflection in beams are a bit tedious and laborious.

When the beam is carrying several point loads.

It is used to find deflection where BM is discontinuous.

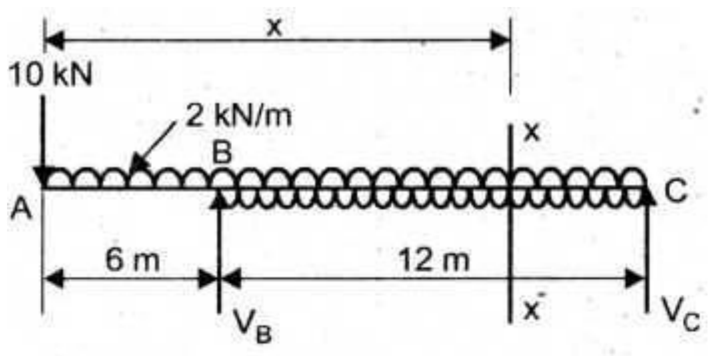
Problems

Let us solve deflection of Macaulay's method



Solution:

Use Macaulay's method,



Extend the UDL up to the right support C and apply upward UDL from B to C of same magnitude (2 KN/m) to compensate.

Support Reactions

Applying $\sum V = 0$ (↑ +)

$V_B + V_C = 10 + (2 \times 6) = 22$ KN

Applying $\sum M_c = 0$ (↻+)

$(V_B \times 12) = (2 \times 6 \times 15) + (10 \times 18)$

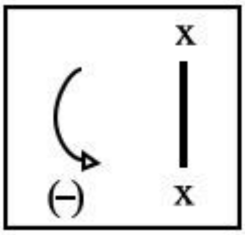
Solving, $V_B = 30$ KN (↑)

∴$V_c = 22 - 30 = -8$ KN $= 8$ KN (↓)

Consider a section XX at a distance of x from the end A

$M_x = V_B(x-6) - 10x + 2(x-6)\frac{(x-6)}{2} - 2\left(x \times \frac{x}{2}\right)$

$$= V_B(x-6) + 2\frac{(x-6)^2}{2} - 10x - \frac{2x^2}{2}$$



$= V_B(x-6) + (x-6)^2 - 10x - x^2$

or $EI\frac{d^2y}{dx^2} = M_x$

$= V_B(x-6) + (x-6)^2 - 10x - x^2$

Integrating once,

$$EI\frac{dy}{dx} = \frac{30}{2}(x-6)^2 + \frac{(x-6)^3}{3} - \frac{10x^2}{2} - \frac{x^3}{3} + C_1$$

Slope equation(∵ $V_B = 30$)

Integrating again,

$$EIy = \frac{15(x-6)^3}{3} \quad \frac{(x-6)^4}{12} - \frac{10x^3}{6} - \frac{x^4}{12} + C_1x + C_2$$

$$\text{or } EI\,y = 5(x-6)^3 + \frac{(x-6)^4}{12} - \frac{10x^3}{6} - \frac{x^4}{12} + C_1x + C_2$$

To Find the constants

at $x = 6$; $y = 0$

Substituting these values in the above equation,

$$0 = \frac{-10\times 6^3}{6} - \frac{6^4}{12} + (C_1\times 6) + C_2$$

$0 = -360 - 108 + 6C_1 + C_2$

or $468 = 6C_1 + C_2$...(i)

at $x = 18$ m; $y = 0$

Substituting these values in the above equation,

$$0 = 5(18-6)^3 + \frac{(18-6)^4}{12} - \frac{10}{6}(18)^3 - \frac{18^4}{12} + (C_1\times 18) + C_2$$

$= 8640 + 1728 - 9720 - 8748 + 18C_1 + C_2$

$= -8100 + 18C_1 + C_2$

or$8100 = 18C_1 + C_2$....(ii)

Solve equations (i) and (ii)

$468 = 6C_1 + C_2$

$8100 = 18C_1 + C_2$

$-7632 = -12C_1$

$\therefore C_1 = 636$

Substituting, $C_2 = -3348$

$$\therefore \text{EI y} = 5(x-6)^3 + \frac{(x-6)^4}{12} - \frac{10x^3}{6} - \frac{x^4}{12} + 636x - 3348$$

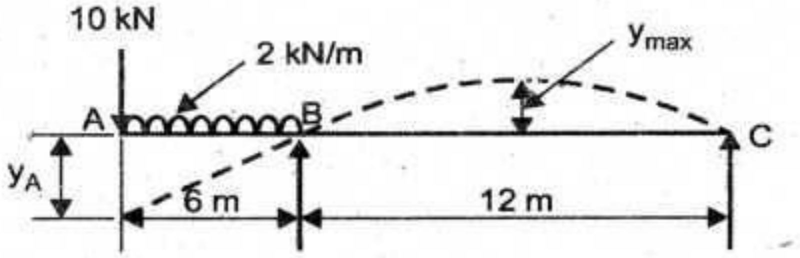
i) Maximum Downward deflection

Occurs at Free end A. Hence substitute x = 0 in the above equation,

EI y , = –3348

or $$y_A = \frac{-3348}{EI} = \frac{-3348 \times 10^{12}}{40000 \times 10^9} = 83.7 \text{ mm}$$

ii) Maximum upward deflection



The deflected shape of beam is shown in Figure above. The maximumupward deflection occurs in the region BC, i.e., 6 < × < 12.

Location of Maximum deflection (i.e., zero slope)

Using slope equation,

$$EI\frac{dy}{dx} = 15(x-6)^2 + \frac{(x-6)^3}{3} - 5x^2 - \frac{x^3}{3} + C_1$$

i.e.,

$$0 = 15(x-6)^2 + \frac{(x-6)^3}{3} - 5x^2 - \frac{x^3}{3} + 636$$

Solve this equation by trial and error,

at x = 12m,LHS = 0,RHS = –48

at x = 13m,LHS = 0,RHS = –91.67

at x = 12.5m,LHS = 0,RHS = –70.96

at x = 11.5m,LHS = 0,RHS = –23

at x = 11m,LHS = 0,RHS = 4.01

∴ x lies in between 11 m and 11.5 m

Take x is approximately 11.25 m

Substituting x = 11.25 in deflection equation to find maximum positive deflection,

∴EIy

$$= 5(x-6)^3 + \frac{(x-6)^4}{12} - \frac{10x^3}{6} - \frac{x^4}{12} + 636x - 3348$$

$$\therefore EI\, y_{max} = 5(11.25-6)^3 + \frac{(11.25-6)^4}{12} + \frac{10 \times 11.25^3}{6}$$

$$-\frac{11.25^4}{12} + (636 \times 11.25) - 3348$$

= 723.51 + 63.3 + 2.373 – 1334 + 7155 – 3348

= 3262

$$y_{max} = \frac{3262}{EI}$$

$$= \frac{3262 \times 10^{12}}{40000 \times 10^9} = 81.55 \text{ mm.}$$

We shall discuss the problem In the beam shown below, determine the slope at the left end C and the deflection at 1m from the left end. Take EI = 0.63 MN m^2

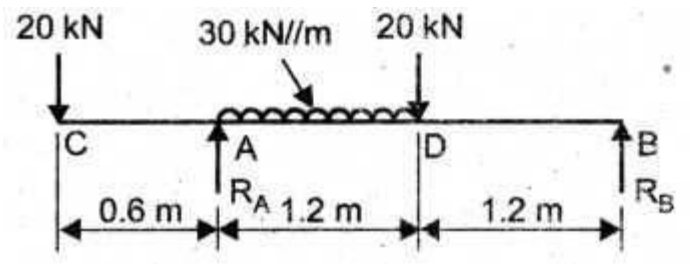
Given:

EI = 0.65 MNm^2

= $0.65 \times 10^6 \times (1000)^2$

= $0.65 \times 10^{12} Nmm^2$.

Solution:



Find θ_C and y at 1 m from the left end.

Support Reactions

Applying $\sum V = 0$

$R_A + R_B = (30 \times 1.2) + 20 + 20 = 76$kN ....(1)

Applying $\sum M_A = 0$

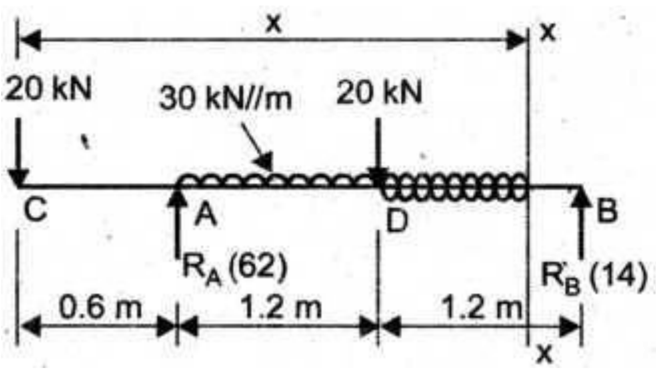
$(R_B \times 2.4) + (20 \times 0.6) = (20 \times 1.2) + \left(30 \times 1.2 \times \frac{1.2}{2}\right)$

$2.4\ R_B + 12 = 24 + 21.6$

SolvingR_B = 14 KN

Substituting R_B in equation (1),

R_A = 62 KN



Macaulay's method

Consider a section XX at a distance of x from C, in the region DB. Extend UDL 30 KN/m upto the section XX and apply the counter UDL in the opposite direction (i.e., upward) from D to the section as shown in figure.

Bending Moment at section XX

$(BM)_x = M = (62 \times (x - 0.6)) - 20(x - 1.8) - 20\,x$

$$-\left\{30(x-0.6)\,x\frac{(x-0.6)}{2}\right\}+\left\{30(x-1.8)\,x\frac{(x-1.8)}{2}\right\}$$

$$EI\frac{d^2y}{dx^2} = M = 62(x-0.6)-20(x-1.8)-20x-15(x-0.6)^2+15(x-1.8)^2.$$

Integrating we get

$$EI\frac{dy}{dx} = \frac{62(x-0.6)^2}{2}-\frac{20(x-1.8)^2}{2}-\frac{20x^2}{2}$$

$$-\frac{15(x-0.6)^3}{3}+\frac{15(x-1.8)^3}{3}+C_1$$

$= 31(x - 0.6)^2 - 10(x - 1.8)^2 - 10x^2 - 5(x - 0.6)^3 + 5(x - 1.8)^3 + C_1$

Again Integrating

$$EI\,y = \frac{31(x-0.6)^3}{3}-\frac{10(x-1.8)^3}{3}-\frac{10x^3}{3}$$

$$-\frac{5(x-0.6)^4}{4}+\frac{5(x-1.8)^4}{4}+C_1x+C_1$$

To find C_1 and C_2

Apply the boundary conditions

at x =0.6m,y = 0

Substituting in above equation,

$$0 = \left(\frac{-10}{3} \times (0.6)^3\right) + (C_1 \times 0.6) + C_2$$

$0.6C_1 + C_2 = 0.72$.... (A)

at x = 3m,y = 0

substituting we get

$$0 = \frac{31}{3}(3-0.6)^3 - \frac{10}{3}(3-1.8)^3 - \frac{10}{3}(3)^3$$

$$-\frac{5}{4}(3-0.6)^4 + \frac{5}{4}(3-1.8)^4 + (3C_1) + C_2$$

$0 = 142.85 – 5.76 – 90 – 41.47 + 2.59 + 3C_1 + C_2$

$0 = 8.21 + 3C_1 + C_2$

$\therefore 3C_1 + C_2 = –8.21$ .... (B)

Solve the equations (A) & (B) $0.6C_1 + C_2 = 0.72$

$$\frac{3C_1 + C_2 = -8.21}{-2.4C_1 = 8.93}$$

$\therefore C_1 = –3.72$

Substituting C_1 in equation (A),

$C_2 = 2.95$

Final Slope Equation is

$$EI\frac{dy}{dx} = 31(x-0.6)^2 - 10(x-1.8)^2 - 10x^2$$

$$- 5(x-0.6)^3 + 5(x-1.8)^3 - 3.72$$

Final Deflection Equation is

$$EI\,y = \frac{31}{3}(x-0.6)^3 - \frac{10}{3}(x-1.8)^3 - \frac{10}{3}x^3$$

$$-\frac{5}{4}(x-0.6)^4 + \frac{5}{4}(x-1.8)^4 - 3.72x + 2.95$$

To find slope at left end C

Substitutex = 0 in Final slope equation

$EI\,\vartheta_c = -3.72$(Note : Negative terms are neglected)

$$\therefore \qquad \theta_C = \frac{-3.72}{EI} = \frac{-3.72 \times 10^9}{0.65 \times 10^{12}}$$

= 0.00572 rad.(anticlockwise slope)

$$= 0.00572 \times \frac{180}{\pi} = 0.327^\circ$$

Deflection at 1 m from left end

Substitute x = 1 m in final deflection equation

$$\therefore \quad EIy = \frac{31}{3}(1-0.6)^3 - \frac{10}{3}(1)^3 - \frac{5}{4}(1-0.6)^4 - (3.72 \times 1) + 2.95$$

(Note: Negative terms are neglected)

= 0.66–3.33 – 0.032 – 3.72 + 2.95

= –3.472

$$\therefore \quad y = \frac{-3.472}{EI} = \frac{-3.472 \times 10^{12}}{0.65 \times 10^{12}} = 5.34 \text{ mm}$$

Problems

Let us solve using Macaulay's method, a beam of length 6 m is simply supported at its ends and carries two point loads of 48 KN and 40 KN at a distance of 1 m and 3 m respectively from the left support.Findi)deflection under each load,ii)maximum deflection and iii)the point at which maximum deflection occurs.

Given E = 2 × 10^5N/mm^2 and I = 85 × $10^6$$mm^4$.

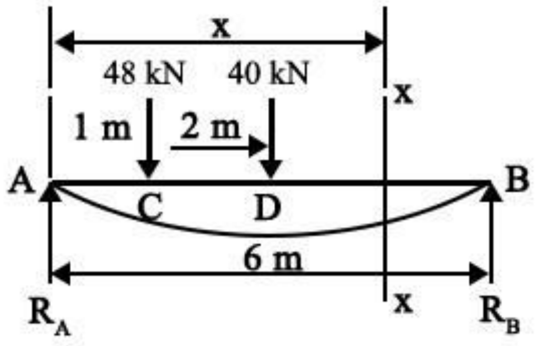
E = 2 × 10^5 N/mm^2; I = 85 × 10^6 mm^4

y_c = ?

y_d = ?

Y_{max} = ?

Solution:



Let R_A and R_B be the support Reactions.

Applying$\sum V = 0(\uparrow=\downarrow)$

$R_A + R_B = 48 + 40 = 88$ KN ....(i)

Applying $\Sigma M_A = 0\ (\circlearrowleft = \circlearrowright)$

$(48 \times l) + (40 \times \mathbf{3}) - (\mathbf{R_B} \times 6) = 0$

Solving,$R_B = 28$ KN

Substituting $R_B = 28$ in equation (1),

$R_A = 60$ KN

Applying Macaulay's method to find deflection

Consider a section XX at a distance of x from the left support a such that it covers all the loads i.e., consider the section XX in DB region as shown in figure.

Using the relation, $EI\frac{d^2y}{dx^2} = M$

Now, $(BM)_x = M = 60x - 48(x - 1) - 40(x - 3)$

$\therefore \quad EI\frac{d^2y}{dx^2} = 60\,x - 48(x-1) - 40(x-3)$

Integrating the above equation,

$$EI\frac{dy}{dx} = \frac{60x^2}{2} - \frac{48(x-1)^2}{2} - \frac{40(x-3)^2}{2} + C_1 \;.....(i)$$

$= 30x^2 - 24(x-1)^2 - 20(x-3)^2$ (Slope equation)

Again integrating the equation (i), we get

$$EI\,y = \frac{30x^3}{3} - \frac{24(x-1)^3}{3} - \frac{20(x-3)^3}{3} + C_1x + C_2 ...(ii)$$

(Deflection equation)

To Find the constants

Apply the Boundary conditions

i) at x = 0; y = 0 and

ii) at x = 6m; y = 0

Substituting x = 0 in equation (ii) $0 = C_2$

Substituting x = 6 in equation (ii)

$$0 = \frac{30}{3}(6)^3 - \frac{24}{3}(6-1)^3 - \frac{20}{3}(6-3)^3 + 6C_1$$

$$0 = \left(\frac{30}{3} \times (6)^3\right) - \left(\frac{24}{3} \times 5^3\right) - \left(\frac{20}{3} \times 3^3\right)$$

$= 2160 - 1000 - 180 + 6C_1$

$0 = 980 + 6C_1$

$C_1 = -163.33$

Now, substituting the values of C_1 and C_2 in equation (ii) we get final deflection equation.

$$\therefore \quad EI\,y = 10x^3 - 8(x-1)^3 - \frac{20}{3}(x-3)^3 - 163.33x$$

i) Deflection under point loads

Deflection under 48 KN (i.e., y_c)

Substitute x = 1m in Final deflection equation

$$EI\,y = 10x^3 - 8\cancel{(x-1)^3} - \frac{20}{3}\cancel{(x-3)^3} - 163.3x$$

orEI $y = 10x^3 - 163.3\,x$

(Note: substituting x = 1 if the value is negative within the brackets (x - 1) and (x - 3), these terms are neglected)

$\therefore$EI yc = 10 × I3 – (163.3 × 1)

= –153.3

$$\therefore \quad y_c = \frac{-153.3}{EI} = \frac{-153.3 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6}$$

= –9.017 mm

(Note that in EIy_c = – 153.3, load in KN and distance is in meter so unit of –153.3 is KN m^3. Converting KN.m^3 into Nmm^3 multiply by $10^3((10^3)^3 = 10^{12})$.

Deflection under 40 KN (i.e., y_D)

Substitute x = 3 m in final deflection equation.

$$EI\,y = 10x^3 - 8(x-1)^3 - \frac{20}{3}\cancel{(x-3)^3} - 163.3x$$

EI y_D = $10x^3 - 8(3 - l)^3 - (163.3 \times 3)$

= 270 – 64 – 489.9

= –283.9

$$\therefore \qquad y_D = \frac{-283.9}{EI} = \frac{-283.9 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6}$$

= – 16.7 mm

To find point of maximum deflection

Deflection is maximum where the slope is zero. Referring the deflected shape of beam, maximum deflection will occur in the region CD i.e., 1 < x < 3.

Consider the slope equation,

i.e., $EI\frac{dy}{dx} = 30x^2 - 24(x-1)^2 - 20(x-3)^2 - 163.3$

Since 1 < x < 3, neglect (x – 3) term

$\therefore \quad EI\frac{dy}{dx} = 30x^2 - 24(x-1)^2 - 163.3$

$0 = 30x^2 - 24(x^2 + 1 - 2x) - 163.3$

$= 30x^2 – 24x^2 – 24 + 48x – 163.3$

$0 = 6x^2 + 48x – 187.3$

$$\therefore \qquad x = \frac{-48 \pm \sqrt{(48)^2 - (4 \times 6 \times (-187.3))}}{2 \times 6}$$

$$= \frac{-48 \pm 82.45}{12}$$

Neglecting Negative value, x = 2.87 m

∴ Maximum deflection occurs at a distance of 2.87 m from the left support.

To find maximum deflection

Substitute the value of x = 2.87m in final deflection equation.

$$EI\ y = 10x^3 – 8(x – l)^3 – \frac{20}{3}(x – 3)^3 – 163.3\ x$$

x = 2.87, so neglecting negative term

$EI\ y = 10x^3 – 8(x – 1)^3 – 163.3\ x$

$EI\ y_{max} = 10 \times (2.87)^3 – 8(2.87 – l)^3 – (163.3 \times 2.87)$

= 236.4 – 52.31 – 468.67

= –284.58

$$y_{max} = \frac{-284.58}{EI} = \frac{-284.58 \times 10^{12}}{2 \times 10^5 \times 85 \times 10^6}$$

= –16.74 mm