Topic: 3. 1 – CURVATURE AND RADIUS OF CURVATURE

Curvature of a curve:

The rate of change of bending of the curve in the given interval is called Curvature of the curve.

Note: (i) The curvature of a straight line is Zero.

(ii) The curvature of a point is infinity.

(iv) The curvature of a circle at any point on it is the same and is equal to the reciprocal of its radius.

Curvature is denoted by $K \left(= \frac{d\psi}{ds} \right)$

Radius of curvature:

The reciprocal of the curvature of a curve at any point is called the radius of curvature at the point and is denoted by ρ.

hence $\rho = \dfrac{1}{\text{curvature}} = \dfrac{ds}{d\psi}$.

Note: the curvature of a circle of radius r at any point is $1/r$.

Cartesian formula for the radius of curvature is $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

1. Find the Radius of curvature of $y = e^x$ at $(0,1)$

solution: Given $y = e^x$.

$$y_1 = \frac{dy}{dx} = e^x \Rightarrow y_{1(0,1)} = e^0 = 1$$

$$y_2 = \frac{d^2y}{dx^2} = e^x \Rightarrow y_{2(0,1)} = e^0 = 1.$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{1} = 2^{3/2}$$

$$\rho = 2\sqrt{2}.$$

2. find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at the point $\left(\frac{a}{4}, \frac{a}{4}\right)$.

solution: Given $\sqrt{x} + \sqrt{y} = \sqrt{a} \longrightarrow (1)$

diff. w.r.t 'x' $\Rightarrow$ $$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = \frac{-\sqrt{y}}{\sqrt{x}}.$$

$$\left(\frac{dy}{dx}\right)_{(a/4, a/4)} = \frac{-\sqrt{a/4}}{\sqrt{a/4}} = -1$$

Again diff. w.r.t 'x'.

$$\frac{d^2y}{dx^2} = -\left[\frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\frac{dy}{dx} - \sqrt{y}\cdot\frac{1}{2\sqrt{x}}}{x}\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{(a/4,\, a/4)} = -\left[\frac{\sqrt{a/4}\cdot\frac{1}{2\sqrt{a/4}}(-1) - \sqrt{a/4}\cdot\frac{1}{2\sqrt{a/4}}}{a/4}\right]$$

$$= -\left[\frac{-\frac{1}{2}-\frac{1}{2}}{a/4}\right] = 4/a.$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+(-1)^2)^{3/2}}{4/a} = \frac{2^{3/2}}{4/a} = \frac{2\sqrt{2}\,a}{4} = \frac{a}{\sqrt{2}}$$

$$\rho = a/\sqrt{2}$$

3. Show that the radius of curvature at $(0, c)$ on the curve $y = c\cosh \frac{x}{c}$ is c.

Solution Given $y = c\cosh \frac{x}{c}$.

diff w.r.t $x \Rightarrow$ $y_1 = \frac{dy}{dx} = c\sinh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$

$$y_{1(0,c)} = 0.$$

$$y_2 = \frac{d^2y}{dx^2} = \cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}$$

$$y_{2(0,c)} = 1/c$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+0)^{3/2}}{1/c}$$

$$\rho = c.$$