Principal stresses and principal planes

Principal plane

The plane which has no shear stress is known as principal plane. These plane carry only normal stresses.

Principal stress

The normal stress acting on a principal plane is known as principal stress.

Problems

Let us see a point with in a body there are two mutually perpendicular stresses of 80 N/mm^2 and 40 N/mm^2 of tensile in nature. Each stress is accompanied by a shear stress of 60 N/mm^2. Determine the normal, shear and resultant stress on an oblique plane at an angle of 45° degree with the axis of the major principal stress.

Given :

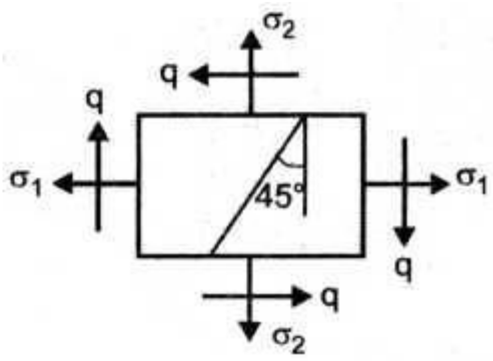
Major principal stress, σ_1 = 80 N/mm^2(tensile)

Minor principal stress, σ_2 = 40N/mm^2(tensile)

Shear stress, q = 60 N/mm^2

Angle of oblique plane with the axis of major principal stress i.e., $(90 - \theta) = 45°$. $\therefore \theta = 45°$

Find:σ_n = ?σ_t = ?σ_R = ?



Solution:

Normal stress (σ_n)

Normal stress, $\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta + q \sin 2\theta$

$$= \left(\frac{80 + 40}{2}\right) + \left(\frac{80 - 40}{2}\right) \cos (2 \times 45) + (60 \sin 2 \times 45)$$

$$= \frac{120}{2} + \frac{40}{2} \cos 90° + 60 \sin 90°$$

= 120 N/mm^2

*Tangential (or shear) stress (**σ_t**)*

Tangential stress, $\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - q \cos 2\theta$

$= \frac{80 - 40}{2} \sin (2 \times 45) - 60 \cos (2 \times 45)$

= 20 sin 90 – 60 cos 90

$= 20\ N/mm^2$

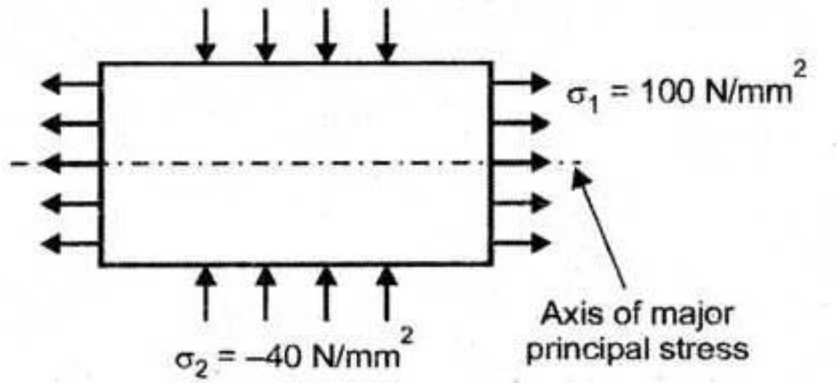
Resultant stress (σ_R)

Resultant stress $\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2}$

$$= \sqrt{(120)^2 + (20)^2}$$

$= 121.655\ N/mm^2$.

We solve a point in a strained material, wherein the principal stresses are 100 N/mm² (tensile) and 40 N/mm² (compressive). Determine analytically the resultant stress in magnitude and direction on a plane inclined at 60° to the axis of major principal stress. What is the maximum intensity of shear stress in the material at that point?



Given :

Major principal stress, $\boldsymbol{\sigma_1}$ = **100** N/mm^2 (tensile)

Minor principal stress, $\sigma_2 = -40\ N/mm^2$ (compressive)

Angle of plane with major principal stress i.e., $(90 - \theta) = 60°$

therefore $\theta = 30°$

Find: σ_R in magnitude and direction.

Solution:

Normal stress,

$$\sigma_n = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\theta$$

$$= \frac{100 + (-40)}{2} + \frac{100 - (-40)}{2} \cos (2 \times 30)$$

= 30 + 70 cos 60

= 65 N/mm^2

Tangential stress,

$$\sigma_t = \frac{\sigma_t - \sigma_2}{2} \sin 20$$

$$= \frac{100 - (-40)}{2} \sin (2 \text{ x } 30)$$

= 60.62 N/mm^2

∴ Resultant stress,

$$\sigma_R = \sqrt{\sigma_n^2 + \sigma_t^2} = \sqrt{(65)^2 + (60.62)^2}$$

= 88.88N/mm^2

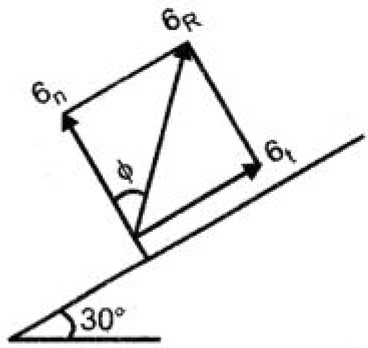
To find inclination of σ_R with normal of inclined plane:

Using the equation,

$$\tan \phi = \frac{\sigma_t}{\sigma_n} \text{ (or) } \phi = \tan^{-1}\left(\frac{\sigma_t}{\sigma_n}\right)$$

$$= \tan^{-1}\left(\frac{60.62}{65}\right)$$

= 43°



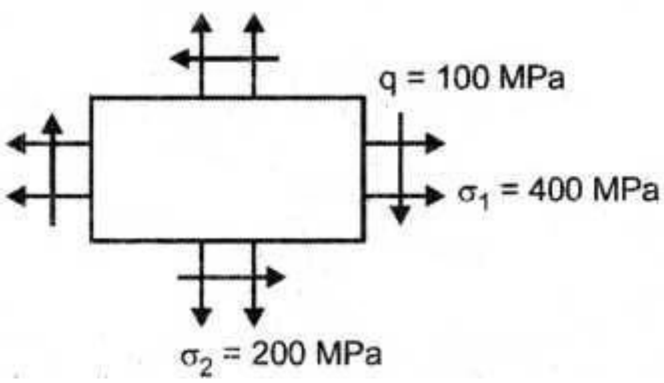
Maximum shear stress:

$$(\sigma_t)_{max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{100-(-40)}{2}$$

= 70 N/mm^2

Let us discuss A plane element in a boiler is subjected to tensile stresses of 400 M Pa on one plane and 200 M Pa on the other at right angles to the former. Each of the above stresses is accompanied by a shear stress of 100 M Pa. Determine the principal stresses and their directions. Also, find maximum shear stress.

Given :



Major tensile stress, σ_1 = 400 M Pa (tensile)

Minor tensile stress, σ_2 = 200 M Pa (tensile)

Shear stressq = 100 M Pa

Find: principal stresses and maximum shear stress.

Solution:

i)Major Principal stress:

Major Principal stress σ_{p1} is given by the equation,

$$\sigma_{p1} = \frac{\sigma_1 + \sigma_2}{2} + \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + q^2}$$

$$= \left(\frac{400+200}{2}\right) + \sqrt{\left(\frac{400-200}{2}\right)^2 + (100)^2}$$

$$= 300 + \sqrt{100^2 + 100^2} = 441.42\ \text{N/mm}^2$$

ii)Minor Principal stress:

Minor principal stress σ_{p2} is given by the equation,

$$\sigma_{p2} = \left(\frac{\sigma_1 + \sigma_2}{2}\right) - \sqrt{\left(\frac{\sigma_1 - \sigma_2}{2}\right)^2 + q^2}$$

$$= \left(\frac{400+200}{2}\right) - \sqrt{\left(\frac{400-200}{2}\right)^2 + (100)^2}$$

$$= 300 - \sqrt{(100)^2 + (100)^2} = 158.78 \text{ N/mm}^2$$

Directions of principal stress:

Using the equation,

$$\tan 2\theta = \frac{2q}{\sigma_1 - \sigma_2} = \frac{2 \times 100}{(400 - 200)} = 1$$

or2θ = $\tan^{-1}$ (1) = 45° (or) 225°

∴ θ = 22°30'and 112°30'

Magnitude of maximum shear stress:

Maximum shear stress is given by the equation,

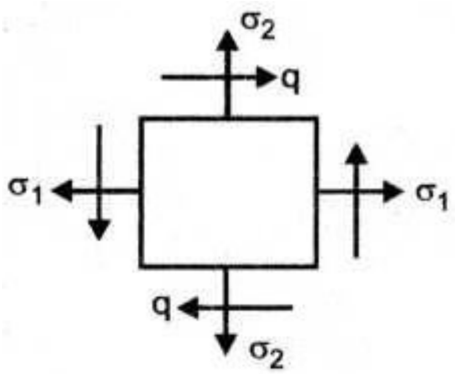
$$(\sigma_t)_{max} = \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4q^2}$$

$$= \frac{1}{2}\sqrt{(400\text{-}200)^2 + 4 \times 100^2} = \frac{1}{2}\sqrt{(200)^2 + (40000)}$$

$$= 141\ 42 \text{ N/mm}^2$$

Here we see an element in plane stress is subjected to stresses σ_1 = 120 N/mm^2 and σ_2 = 45 N/mm^2 (both tensile) and shearing stress of 30 N/mm^2 as shown in figure below. Determine the stresses acting as an element rotated through an angle θ = 45°.

Solution:



Element is rotated through an angle 45°.

∴ $\theta = 45°$

The stresses on the rotated planes are normal stress and shear stress.

Normal stress:

Normal stress on the rotated plane,

$$\sigma_n = \left(\frac{\sigma_1 + \sigma_2}{2}\right) + \left(\frac{\sigma_1 - \sigma_2}{2}\right) \cos 2\theta + q \sin 2\theta$$

$$= \left(\frac{120 + 45}{2}\right) + \left(\frac{120 - 45}{2}\right) \cos 90 + 30 \sin 90$$

$= 82.5 + 0 + 30 = 112.5\ N/mm^2$

Tangential stress:

Tangential stress on the rotated plane,

$$q = \sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - q\cos 2\theta$$

$$= \left(\frac{120-45}{2}\right) \sin 90 - 30\cos 90$$

= 37.5 – 0

= 37.5 N/mm^2

The normal stresses on the other plane, at right angles to the previous one is determined by substituting θ as (90 + 45) = 135° in σ_n equation.

$$\therefore \quad \sigma_{n2} = \left(\frac{\sigma_1 + \sigma_2}{2}\right) + \left(\frac{\sigma_1 - \sigma_2}{2}\right) \cos 2\theta + q \sin 2\theta$$

$$= \left(\frac{120+45}{2}\right) + \left(\frac{120-45}{2}\right) \cos 270° + 30 \sin 270°$$

= 82.5 + 0 + (– 30)

= 52.5 N/mm^2

The stresses on the rotated element through an angle θ = 45° is shown below.