Topic: 1.6 – CAYLEY HAMILTON THEOREM

Cayley - Hamilton Theorem:

Every square matrix satisfies its own Characteristic equation.

Uses of Cayley - Hamilton theorem:

To calculate (i) the positive integral powers of A and
(ii) the inverse of a square matrix A.

Problem:

verify that $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ satisfies its own characteristic equation and hence find A^4.

Soln:

Given $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

The Char. equ. of A is $\lambda^2 - S_1\lambda + S_2 = 0$.

where $S_1 = 1 + (-1) = 0$

$S_2 = |A| = -1 - 4 = -5$

$\therefore$ The Char. equ. is $\lambda^2 - 0\lambda - 5 = 0$

(ie) $\lambda^2 - 5 = 0$

[By C-H, Every square matrix satisfies its own char. equ.]

(ie) To prove: $A^2 - 5I = 0$ ——— (1)

$A^2 = A \times A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

$$= \begin{bmatrix} 1+4 & 2-2 \\ 2-2 & 4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ The given matrix A satisfies its own cha. equ.

To find A^4:

consider $A^2 - 5I = 0$

$\Rightarrow A^2 = 5I$

Multiply A^2 on both sides

$$A^4 = A^5(5I)$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$$

Find A^{-1} if $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley-Hamilton theorem.

Soln.

The char. eqn. of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where $S_1 = 1+2-1 = 2$

$$S_2 = \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$$= (-2+1) + (-1-8) + (2+3)$$

$$= -1 + (-9) + 5$$

$$= -5$$

$$S_3 = |A| = 1(-2+1) + 1(-3+2) + 4(3-4)$$

$$= 1(-1) + 1(-1) + 4(-1)$$

$$= -1-1-4 = -6$$

$\therefore$ The cha eqn. is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$.

By Cayley Hamilton theorem,

Every square matrix satisfies its own char. eqn.

$$\therefore A^3 - 2A^2 - 5A + 6I = 0 \quad \text{——} \ (1)$$

To find A^{-1}

$(1) \times A^{-1} \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$.

$$A^2 - 2A - 5I + 6A^{-1} = 0$$

$$6A^{-1} = -A^2 + 2A + 5I$$

$$A^{-1} = \frac{1}{6}\left[-A^2 + 2A + 5I\right] \quad \text{——} (3)$$

$$A^2 = A \times A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (3) $\Rightarrow$ $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

If $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, find A^n interms of A.

Solu: Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$.

The Char. equ. of A is $\lambda^2 - S_1\lambda + S_2 = 0$

where $S_1 = 1+2 = 3$

$S_2 = |A| = 2 - 0 = 2$

$\therefore$ The Char. equ. is $\lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow (\lambda-2)(\lambda-1) = 0$

$\lambda = 2, \lambda = 1$

Hence the Eigenvalues of A are $1, 2$.

To find A^n

When λ^n is divided by $\lambda^2-3\lambda+2$

let the quotient be $Q(\lambda)$ and remainder be $a\lambda+b$

$$\lambda^n = (\lambda^2-3\lambda+2)Q(\lambda)+a\lambda+b \quad \text{---} (1)$$

When $\lambda=1$ | When $\lambda=2$

(1) $\Rightarrow 1^n = a+b$ | (1) $\Rightarrow 2^n = 2a+b$

$$2a+b = 2^n \quad \text{———} (2)$$

$$a+b = 1^n \quad \text{———} (3)$$

Solving (2) & (3) we get

(2) $-$ (3) $\Rightarrow a = 2^n - 1^n$

(2) $- 2\times$(3) $\Rightarrow b = -2^n + 2(1^n)$

(ie) $a = 2^n - 1^n$

$b = 2(1^n) - 2^n$

Replacing λ by the matrix A in (1), $A^n = (A^2-3A+2I)Q(A)+aA+bI$

[By C-H $A^2-3A+2I = 0$]

(1) $\Rightarrow A^n = aA+bI$

$$A^n = (2^n-1^n)\begin{bmatrix}1 & 2\\ 0 & 2\end{bmatrix} + [2(1)^n - 2^n]\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$