Topic: 2.8 – Alternating series – Leibnitz's test

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Alternating Series - Leibnitz's test:

Alternating series.

A series in which terms are alternately positive & (or) negative is called an alternating serie. Eg: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \cdots$

Leibnitz's rule.

An alternating series $u_1 - u_2 + u_3 - u_4 + \cdots$

Converges if (i) $u_n - u_{n-1} < 0$ (consider numerical value of u_n & u_{n-1})

(ii) $\lim_{n\to\infty} u_n = 0$ (consider numerical values of u_n)

1. Discuss the convergence of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Soln: The terms of the given series are alternatly +ve & -ve.

Step:1 To find $u_n = \frac{1}{n}$

$$u_{n-1} = \frac{1}{n-1}$$

$$u_n - u_{n-1} = \frac{1}{n} - \frac{1}{n-1}$$

$$= \frac{n-1-n}{n(n-1)} = \frac{-1}{n(n-1)} < 0$$

$$\Rightarrow u_n - u_{n-1} < 0 \quad \text{---} (1)$$

Step:2 To find $\lim_{n\to\infty} u_n$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \text{---} (2)$$

Step:3 Conclusion:

From (1) & (2) Leibnitz's rule satisfied.

$\therefore$ The given series is convergent

2. Discuss the convergence of the series $1-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+ - - -$

Soln:

The terms of the given series are alternately +ve & −ve.

Step:1 To find u_n & u_{n-1} (numerically)

Here $u_n = \frac{1}{\sqrt{n}}$

$u_{n-1} = \frac{1}{\sqrt{n-1}}$

$u_n - u_{n-1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}} = \frac{\sqrt{n-1}-\sqrt{n}}{\sqrt{n}\sqrt{n-1}} < 0$

$\Rightarrow u_n - u_{n-1} < 0$ —— (1)

Step:2 To find $\lim_{n\to\infty} u_n$

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ —— (2)

Step:3: from (1) & (2) Leibnitz's rule satisfied

∴ the given series is convergent.

3. Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2n-1}$

Soln:

The terms of the given series are alternately +ve and −ve.

Here $u_n = \frac{n}{2n-1}$, $u_{n-1} = \frac{n-1}{2(n-1)-1} = \frac{n-1}{2n-3}$

$u_n - u_{n-1} = \frac{n}{2n-1} - \frac{n-1}{2n-3}$

$= \frac{2n^2-3n-2n^2+n+2n-1}{(2n-1)(2n-3)} = \frac{-1}{(2n-1)(2n-3)} < 0$ — (1)

$\Rightarrow u_n - u_{n-1} < 0$.

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{1}{2-1/n} = \frac{1}{2} \neq 0$ — (2)

From (1) & (2) Leibnitz's rules are not satisfied.

∴ The given series is not convergent.

⇒ Hence the given series is oscillatory.

4. Examine the character of the series $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1} = \sum \frac{(-1)^n}{n^2+1}$

Soln:

The terms of the given series are alternately +ve and negative.

Here $u_n = \frac{1}{n^2+1}$

$$u_{n-1} = \frac{1}{(n-1)^2+1} = \frac{1}{n^2-2n+1+1}$$

$$= \frac{1}{n^2-2n+2}$$

$$u_n - u_{n-1} = \frac{1}{n^2+1} - \frac{1}{n^2-2n+2}$$

$$= \frac{n^2-2n+2-n^2-1}{(n^2+1)(n^2-2n+2)}$$

$$= \frac{-2n+1}{(n^2+1)(n^2-2n+2)} < 0$$

$$\Rightarrow u_n - u_{n-1} < 0 \quad \text{—(1)}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0 \quad \text{—(2)}$$

from (1) & (2) Leibnitz's rules are satisfied

$\therefore$ the given series is convergent.

5. Discuss the convergence of the series

$$\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \dots \quad 0 < x < 1.$$

Soln:

The terms of the given series are alternately positive and negative.

Here $u_n = \frac{x^n}{x^n+1}$; $u_{n-1} = \frac{x^{n-1}}{1+x^{n-1}}$

$$u_n - u_{n-1} = x^n \left[\frac{1}{1+x^n} - \frac{1}{x+x^{n+1}} \right]$$

$$= x^n \left[\frac{x + x^n - 1 - x^n}{(1+x^n)(x+x^n)} \right]$$

$$= x^n \left[\frac{x-1}{(1+x^n)(x+x^n)} \right] < 0 \quad [\because 0<x<1]$$

$$\Rightarrow u_n - u_{n-1} < 0 \text{ --- } (1)$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0 \text{ --- } (2)$$

$[\because x<1, \ x^n \to 0 \text{ as } n\to\infty]$

from (1) & (2) Leibnitz's rule satisfied.

$\therefore$ the given series is convergent.

6. Discuss the convergence of the series

$$\left(\frac{1}{2} - \frac{1}{\log 2}\right) - \left(\frac{1}{2} - \frac{1}{\log 3}\right) + \left(\frac{1}{2} - \frac{1}{\log 4}\right) \cdots$$

Soln: The terms of the given series are alternately positive and negative

Here $u_n = \frac{1}{2} - \frac{1}{\log(n+1)}$, $u_{n-1} = \frac{1}{2} - \frac{1}{\log(n-1+1)} = \frac{1}{2} - \frac{1}{\log n}$

$$u_n - u_{n-1} = \left[\frac{1}{2} - \frac{1}{\log(n+1)}\right] - \left[\frac{1}{2} - \frac{1}{\log n}\right]$$

$$= \frac{1}{\log n} - \frac{1}{\log(n+1)} = \frac{\log(n+1) - \log(n)}{\log n \log(n+1)} \nless 0$$

$$\Rightarrow u_n - u_{n-1} \nless 0 \text{ --- } (1)$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left[\frac{1}{2} - \frac{1}{\log(n+1)}\right] = \frac{1}{2} \neq 0 \text{ --- } (2)$$

from (1) & (2) both the condition of Leibnitz's rule not satisfied.

Hence the series is oscillatory and it oscillates b/w $-\infty$ and ∞.