Topic: 2.5 – Integral test

Integral test:

Cauchy's integral test:

If $\sum u_n$ is a series of positive terms and if $u_x = f(x)$ be such that

(i) $f(x)$ is continuous in $1 < x < \infty$.

(ii) $f(x)$ decreases as x increases then the series $\sum u_n$ is convergent (or) divergent according as the integral $\int_1^\infty f(x)\,dx$ finite or infinite.

Working Procedure:

1. Find $u_n = f(n)$ [General term]
   Change 'n' to x $\Rightarrow u_x = f(x)$
2. Ensure that $f'(x) < 0$.
3. Evaluate $\int_1^\infty f(x)\,dx$
4. Conclusion
   $\int_1^\infty f(x)\,dx$ = finite $\Rightarrow \sum u_n$ is convergent.
   $\int_1^\infty f(x)\,dx$ = infinite $\Rightarrow \sum u_n$ is divergent

1. Use integral test to discuss the nature of convergence of the series $\frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \cdots$

Soln: Given: $\sum u_n = \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \frac{1}{4\cdot5} + \cdots$

Step: 1 To find u_x

Here $u_n = \frac{1}{n(n+1)}$

$\Rightarrow f(x) = u_x = \frac{1}{x(x+1)} = \frac{1}{x^2+x}$

Step: 2

$$f'(x) = \frac{-(2x+1)}{(x^2+x)^2} < 0$$

Hence, $f(x)$ is decreasing.

Step: 3: Evaluate $\int_1^\infty f(x)dx$

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{1}{x(x+1)}dx$$

$$= \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \left[\log x - \log(x+1)\right]_1^\infty$$

$$= \left[\log \frac{x}{x+1}\right]_1^\infty$$

$$= \left[\log \frac{1}{1+1/x}\right]_1^\infty$$

$$= \log 1 - \log 1/2$$

$$= 0 - \log 1/2$$

$$= -\log 1/2 = \text{finite}$$

Hence, Σu_n is convergent.

2. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Soln:

~~Let $f(x) = \frac{1}{x^2+1}$~~

Step: 1 To find u_x

Here $u_n = \frac{1}{n^2+1}$

$\Rightarrow f(x) = u_x = \frac{1}{x^2+1}$

Step: 2:

$f'(x) = -\frac{2x}{(x^2+1)^2} < 0$

Hence, $f(x)$ is $\downarrow$ing.

Step: 3 :

$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{dx}{x^2+1}$

$= \left[\tan^{-1}(x)\right]_1^{\infty}$

$= \tan^{-1}(\infty) - \tan^{-1}(1)$

$= \frac{\pi}{2} - \pi/4$

$= \pi/4$ (finite)

$\therefore \int_1^{\infty} f(x)\,dx$ converges to $\pi/4$

$\therefore \sum u_n$ is convergent

4. Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\log n}$

Soln.

Let $f(x) = \frac{1}{x\log x}$ for $x \geq 2$

Then $\sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n\log n}$

$f(x) > 0$ & $f(x)$ is $\downarrow$ in $[2, \infty)$

$$\int_1^{\infty} f(x)\,dx = \int_2^{\infty} \frac{1}{x\log x}\,dx$$

put $\log x = t$, $\frac{1}{x}dx = dt$ | $x=2$, $t=\log 2$; $x=\infty$, $t=\infty$

$$= \int_{\log 2}^{\infty} \frac{dt}{t}$$

$$= [\log t]_{\log 2}^{\infty}$$

$$= \log\infty - \log(\log 2)$$

$$= \infty$$

$\therefore \int_2^{\infty} f(x)\,dx$ is divergent in $[2, \infty)$

By integral test, $\sum_{n=2}^{\infty} f(n)$ is also divergent.

5. Show that the series $e^{-1} + 2e^{-2} + 3e^{-3} + \dots + ne^{-n} + \dots$ converges.

Soln.

Let $f(x) = xe^{-x}$

Then $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} ne^{-n}$

$f(x) > 0$ & is $\downarrow$ing in $[1, \infty)$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} xe^{-x}\,dx$$

$$= [xe^{-x} - e^{-x}]_1^{\infty}$$

$\left[\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^{x}} = 0 \text{ \& } \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^{x}} = 0\right]$

$$= 0 + e^{-1} + 0 + e^{-1} = 2e^{-1}$$

$$= 2/e \text{ (finite)}$$

$\therefore \int_1^{\infty} f(x)\,dx$ converges. By integral test $\sum_{n=1}^{\infty} f(n)$ also converges.

6. Test the convergence of the series $\sin\pi + \frac{1}{4}\sin\frac{\pi}{2} + \frac{1}{9}\sin\frac{\pi}{3} + \ldots$

Soln:

Let $f(x) = \frac{1}{x^2}\sin\frac{\pi}{x}$

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n^2}\sin\frac{\pi}{n}$$

$$\int_1^{\infty} f(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\sin\frac{\pi}{x}\,dx$$

Put $\frac{\pi}{x} = t$, $-\frac{\pi}{x^2}dx = dt$ | $x = 1,\ t = \pi$; $x = \infty,\ t = 0$

$$= \frac{1}{\pi}\int_{\pi}^{0} \sin t\,(-dt)$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin t\,dt$$

$$= \frac{1}{\pi}\left[-\cos t\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\cos\pi + \cos 0\right]$$

$$= \frac{1}{\pi}\left[1 + 1\right]$$

$$= 2/\pi \text{ (finite)}$$

$\therefore \int_1^{\infty} f(x)\,dx$ converges.

By integral test

$\sum_{n=1}^{\infty} f(n)$ is also converges.