Topic: 1.2 – EIGEN VALUES AND EIGEN VECTORS

Eigen-values (or) Proper values (or) Latent roots (or) Characteristic roots:

Let $A = [a_{ij}]$ be square matrix. The Characteristic equ. of A is $|A - \lambda I| = 0$. The roots of the characteristic equ. are called Eigen values of A.

Eigen Vector (or) Latent Vector:

Corresponding to each characteristic root λ, there corresponds non-zero vector X which satisfies the equ. $(A - \lambda I)X = 0$. The non-zero vector X are called characteristic vectors (or) eigen vectors.

Working Rule to find Eigen values & Eigenvectors:

Step 1: To find the characteristic equ. $|A - \lambda I| = 0$.

Step 2: To solve the characteristic equ. we get characteristic roots. They are called Eigen values.

Step 3: To find Eigen vectors, solve $(A - \lambda I)X = 0$ for the different values of λ.

Find the Eigenvalues and Eigen vectors of the matrix $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Soln:

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Step 1: To find the Characteristic equ.

The Chara. equ. of A is $\lambda^2 - S_1\lambda + S_2 = 0$.

where S_1 = Sum of main diagonal elements

$= 1 + (-1) = 0$.

$S_2 = |A| = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4$

$\therefore$ The Char. equ. is $\lambda^2 - 0\lambda - 4 = 0$.

(ie) $\lambda^2 - 4 = 0$

Step: 2: To solve the Char. equ.

$\lambda^2 = 4$

$\lambda = \pm 2$

Hence the Eigenvalues are $-2, 2$

Step 3: To find the Eigen vectors solve $(A-\lambda I)x = 0$

$$\left[\begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—} (1)$$

Case (i):

If $\lambda = -2$ then (1) becomes

$$\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 = 0$$
$$3x_1 + x_2 = 0$$

We get only one eqn. $3x_1 + x_2 = 0$

$$3x_1 = -x_2$$
$$\frac{x_1}{1} = +\frac{x_2}{-3}$$

∴ The corresponding Eigenvector is

$$X_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Case (ii)

If $\lambda = 2$ then eqn. (1) become.

$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$
$$3x_1 - 3x_2 = 0$$

(ie) we get only one eqn. $x_1 - x_2 = 0$

$$x_1 = x_2$$

(ie) $\frac{x_1}{1} = \frac{x_2}{1}$

Hence the corresponding Eigen vector is $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Result:

1. Eigen values of A are $(-2, 2)$
2. Eigen vectors : $\lambda = -2 \Rightarrow X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

   : $\lambda = 2 \Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Find the Eigenvalues & Eigenvectors of

$$A = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix}$$

Soln:

Step 1: To find the Charac. eqn.

The Charac. eqn. of A is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = Sum of its leading diagonal elements

$= 7 + 6 + 5 = 18$

S_2 = Sum of the minors of its leading diagonal.

$$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix}$$

$$= (30-4) + (35-0) + (42-4)$$

$$= 26 + 35 + 38 = 99$$

$$S_3 = |A| = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix}$$

$$= 7[30-4] + 2[-10-0] + 0]$$

$$= 7(26) - 20$$

$$= 182 - 20 = 162$$

$\therefore$ The charac. eqn is

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0.$$

Step: 2 To solve the charac. eqn.

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0 \quad \text{—(1)}$$

If $\lambda = 1$ then (1) $\Rightarrow 1 - 18 + 99 - 162 \neq 0$.

If $\lambda = -1$ then (1) $\Rightarrow -1 - 18 - 99 - 162 \neq 0$.

If $\lambda = 2$ then (1) $\Rightarrow 8 - 72 + 198 - 162 \neq 0$

If $\lambda = -2$ then (1) $\Rightarrow -8 - 72 - 198 - 162 \neq 0$.

If $\lambda = 3$ then $27 - 162 + 297 - 162 = 0$

$\therefore \lambda = 3$ is a root

By synthetic Division

$$\begin{array}{c|cccc} 3 & 1 & -18 & 99 & -162 \\ & 0 & 3 & -45 & 162 \\ \hline & 1 & -15 & 54 & \boxed{0} \end{array}$$

$$(\lambda-3)(\lambda^2-15\lambda+54)=0.$$

$$(\lambda-3)(\lambda-6)(\lambda-9)=0.$$

$$\Rightarrow \lambda=3,\ \lambda=6\ \&\ \lambda=9.$$

Step: 3: Solve $(A-\lambda I)X=0$.

$$\begin{bmatrix} 7-\lambda & -2 & 0 \\ -2 & 6-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——} (A)$$

Case (i)

If $\lambda=3$, (A) $\Rightarrow$

$$4x_1-2x_2+0x_3=0 \quad \text{——} (2)$$

$$-2x_1+3x_2-2x_3=0 \quad \text{——} (3)$$

$$0x_1-2x_2+2x_3=0 \quad \text{——} (4)$$

Solving (3) & (4)

$$\begin{array}{cccccc} & & x_1 & x_2 & x_3 & \\ -2 & 3 & -2 & -2 & 3 & \\ 0 & -2 & 2 & 0 & -2 & \end{array}$$

$$\frac{x_1}{6-4}=\frac{x_2}{0+4}=\frac{x_3}{4-0}$$

$$\frac{x_1}{2}=\frac{x_2}{4}=\frac{x_3}{4}$$

$$\frac{x_1}{[illegible]}=\frac{x_2}{[illegible]}=\frac{x_3}{[illegible]}$$

Hence the corresponding Eigen vector is $X_1 = \begin{bmatrix}1\\2\\2\end{bmatrix}$

Case (ii): When $\lambda = 6$. (A) become.

$x_1 - 2x_2 + 0x_3 = 0$ —— (5)

$-2x_1 + 0x_2 - 2x_3 = 0$ —— (6)

$0x_1 - 2x_2 - x_3 = 0$ —— (7)

Solving (6) & (7)

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & \\ -2 & 0 & -2 & -2 & 0 \\ 0 & -2 & -1 & 0 & -2 \end{array}$$

$$\frac{x_1}{0-4} = \frac{x_2}{0-2} = \frac{x_3}{4-0}$$

$$\frac{x_1}{-4} = \frac{x_2}{-2} = \frac{x_3}{4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

Hence the Eigenvector is $X_2 = \begin{bmatrix}2\\1\\-2\end{bmatrix}$

Case (iii) When $\lambda = 9$, (A) become.

$-2x_1 - 2x_2 + 0x_3 = 0$ —— (8)

$-2x_1 - 3x_2 - 2x_3 = 0$ —— (9)

$0x_1 - 2x_2 - 2x_3 = 0$ —— (10)

Solving (9) & (10) we get

$$\begin{array}{ccccc} & x_1 & x_2 & x_3 & \\ -2 & -3 & -2 & -2 & -3 \\ 0 & -2 & -4 & 0 & -2 \end{array}$$

$$\frac{x_1}{12-4} = \frac{x_2}{0-8} = \frac{x_3}{4-0}$$

$$\frac{x_1}{8} = \frac{x_2}{-8} = \frac{x_3}{4}$$