Topic: 2.9 – Absolute and Conditional Convergence

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SERIES OF POSITIVE AND NEGATIVE TERMS – ABSOLUTE AND CONDITIONAL CONVERGENCE:

Absolutely convergent:

A Series Σu_n which contains positive as well as negative terms is said to be absolutely convergent if $\Sigma |u_n|$ is convergent.

Eg:

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is absolutely convergent

Since the series of absolute terms

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is known to be convergent.

Conditionally convergent:

If $\Sigma |u_n|$ is divergent but Σu_n is convergent, then Σu_n is said to be conditionally convergent.

Eg:

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ is convergent

the series of absolute values

$1 + 1/2 + 1/3 + 1/4 + \cdots$ is divergent.

$\therefore$ $1 - 1/2 + 1/3 - 1/4 + \cdots$ is conditionally convergent

1. Show that the series $1+\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}- - - -$ is absolutely convergent.

Soln: Given $\Sigma u_n = 1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}+ - - -$

$\Rightarrow \Sigma |u_n| = 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+ - - -$

To prove: Σu_n is absolutely convergent.

(ie) To prove: $\Sigma |u_n|$ is convergent.

$\Sigma |u_n| = \Sigma \frac{1}{n^2}$ is of the form $\Sigma \frac{1}{n^p}$ where $p=2>1$

$\Rightarrow \Sigma |u_n|$ is convergent

$\Rightarrow \Sigma u_n$ is absolutely convergent.

2. Prove that the alternating series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+ - -$ is conditionally convergent.

Soln: Given $\Sigma u_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+ - - -$

$\Rightarrow \Sigma |u_n| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+ - - -$

To prove: Σu_n is conditionally convergent

(ie) To prove: (1) Σu_n is convergent.

(2) $\Sigma |u_n|$ is divergent.

(1) $\Sigma u_n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+ - - -$

Given is a alternating series.

Here $u_n = \frac{1}{n}$

$u_{n-1} = \frac{1}{n-1}$

$u_n - u_{n-1} = \frac{1}{n}-\frac{1}{n-1} = \frac{n-1-n}{n(n-1)} = \frac{-1}{n(n-1)} < 0$

(ie) $u_n - u_{n-1} < 0$ ——— (1)

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ——— (2)

From ① & ② by Leibnitz's rule satisfied.

$\therefore$ the given series is convergent.

$\Sigma |u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

$\Rightarrow \Sigma \frac{1}{n}$ is of the form

$\Sigma \frac{1}{n^p}$ where $p = 1$

$\Rightarrow \Sigma |u_n|$ diverges.

$\therefore \Sigma u_n$ is conditionally convergent.

3. Test for conditional convergence of the following series $\frac{1}{2^3} - \frac{1}{3^3}(1+2) + \frac{1}{4^3}(1+2+3) - \frac{1}{5^3}(1+2+3+4) + \cdots \infty$

Soln: Given: $\Sigma u_n = \Sigma \frac{1}{(n+1)^3}[1+2+3+\cdots n](-1)^{n-1}$

$= \Sigma \frac{(-1)^{n-1} \frac{n(n+1)}{2}}{(n+1)^3}$

$= \Sigma \frac{1}{2} \frac{n}{(n+1)^2} (-1)^{n-1}$

$\Rightarrow \Sigma |u_n| = \Sigma \frac{1}{2} \frac{n}{(n+1)^2}$

(i) To find $\Sigma |u_n|$ is convergent (or) divergent

Apply order test

$u_n = \frac{1}{2} \frac{n}{(n+1)^2} = \frac{1}{n^k}$ where $k = p - q = 2 - 1 = 1$

$\therefore \Sigma |u_n|$ is divergent.

(ii) To find Σu_n is convergent (or) divergent.

Apply Leibnitz's test

$u_n = \frac{1}{2} \frac{n}{(n+1)^2}$, $u_{n-1} = \frac{1}{2} \frac{n-1}{n^2}$

$u_n - u_{n-1} = \frac{1}{2} \left[\frac{-n^2 + n + 1}{n^2(n^2+2n+1)} \right] < 0, (n > 1)$ — ①

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2} \frac{n}{(n+1)^2} = 0$

$\therefore$ the gn. series is converged

$\Rightarrow \Sigma u_n$ is conditionally converged.

4. Test the convergence and absolute convergence of the series $\frac{1}{\sqrt{2}+1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}+1} - \frac{1}{\sqrt{5}+1} + \cdots$

Soln:

Given : $\Sigma u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n+1}+1} \Rightarrow \Sigma |u_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+1}$

(i) To find $\Sigma|u_n|$ is convergent (or) not.

Apply order test.

$u_n = \frac{1}{n^{p-q}} = \frac{1}{n^k} = \frac{1}{n^{1/2}}$ $\therefore k < 1$

$\therefore \Sigma|u_n|$ is diverged

(ii) To find Σu_n is convergent (or) not.

Apply Leibnitz's test

$u_n = \frac{1}{\sqrt{n+1}+1}$ $u_{n-1} = \frac{1}{\sqrt{n}+1}$

$u_n - u_{n-1} = \frac{1}{\sqrt{n+1}+1} - \frac{1}{\sqrt{n}+1} = \frac{\sqrt{n} - \sqrt{n+1}}{(\sqrt{n+1}+1)(\sqrt{n}+1)} < 0$

$u_n - u_{n-1} < 0$ —— (1)

$\lim_{n\to\infty} u_n = \frac{1}{\sqrt{n+1}+1} = 0$ —— (2)

From (1) & (2) the given series is convergent

$\Rightarrow \Sigma u_n$ is conditionally converged.

5. Test $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2}$ for convergence and absolute convergence.

Soln: Given $\Sigma u_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\log n)^2} \Rightarrow \Sigma|u_n| = \sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$

To find $\Sigma|u_n|$ is convergent (or) not

$u_n = \frac{1}{n^{p-q}} = \frac{1}{n^3}$ where $p = 3 > 1$

$\therefore \Sigma|u_n|$ is convergent.

$\Rightarrow \Sigma u_n$ is absolutely convergent.

6. Show that the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{\sqrt{2n+1}}$ is absolutely convergent for $|x|<1$, conditionally convergent for $x=1$ & divergent for $x=-1$.

Soln:

$$\sum u_n = \frac{(-1)^{n-1} x^n}{\sqrt{2n+1}}, \quad \sum |u_n| = \frac{x^n}{\sqrt{2n+1}}$$

To find $\Sigma |u_n|$ convergence (or) not.

Apply ratio test:

$$u_{n+1} = \frac{x^{n+1}}{\sqrt{2n+3}}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{n+1}}{\sqrt{2n+3}} \cdot \frac{\sqrt{2n+1}}{x^n} = \frac{x\sqrt{2+1/n}}{\sqrt{2+3/n}}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |x|$$

If $|x|<1$ then $\Sigma |u_n|$ is convergent.

$\Rightarrow \Sigma u_n$ is absolutely convergent.

If $x=1 \Rightarrow \Sigma u_n = \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} + \ldots$ which is convergent.

If $x=-1 \Rightarrow \Sigma u_n = -\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} - \ldots$

$\Rightarrow \Sigma u_n$ is divergent.

If $x=1$, $\Sigma |u_n|$ is divergent & Σu_n is convergent

$\Rightarrow \Sigma u_n$ is conditionally convergent for $x=1$.