Topic: 3.8 – ENVELOPES

Envelope:

A curve which touches each member of a family of curve is called the envelope of that family of curves,

The envelope of a family of curves is the locus of the ultimate points of intersection of the consecutive members of the family.

Method 1: for finding envelope:

1) If the family of curves is expressed as a quadratic equation of the parameter, say, $A\lambda^2+B\lambda+C=0$ where A, B, C are functions of x and y and λ is the parameter then the envelope of this family is given by $B^2-4AC=0$

2) Analytic method to find the Envelope of the family of curves.

1. Differentiate $f(x,y,c)=0$ partially w.r.t the parameters c.

2. Eliminate 'c' from $f(x,y,c)=0$ & $\frac{\partial}{\partial c}f(x,y,c)=0$

We get the envelope of the family.

Problems:

1. Find the envelope of the family of straight lines $y = mx + am^2$; m being the parameter.

Soln: Given $y = mx + am^2$,

$$am^2 + mx - y = 0.$$

This is quadratic in m, so the envelope is $B^2 - 4AC = 0$, here $A = a$, $B = x$, $C = -y$

$$x^2 - 4a(-y) = 0$$

$$\Rightarrow x^2 + 4ay = 0$$

2. Find the envelope of the family of lines $y = mx + a/m$ where 'a' is a constant.

Soln: Given $y = mx + a/m$

$$y = \frac{m^2x + a}{m}.$$

$$my = m^2x + a.$$

$$m^2x - my + a = 0.$$

This is a quadratic in 'm'.

So the envelope is $B^2 - 4AC = 0$

$$(-y^2) - 4xa = 0$$

$$\text{(i.e)}\ y^2 = 4ax.$$

3. Find the envelope of the family of straight lines. $x\cos\theta + y\sin\theta = a$, where θ being parameter

soln:

Given $x\cos\theta + y\sin\theta = a \rightarrow (1)$

(1) diff p.w.r.t $\Rightarrow -x\sin\theta + y\cos\theta = 0 \rightarrow (2)$

Squaring and adding (1) + (2) the envelope

$$x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) = a^2$$

(ie) $x^2 + y^2 = a^2$, which is a circle.

4. Find the envelope of the family of straight lines $y = mx + \sqrt{a^2m^2+b^2}$ where 'm' is the parameter.

soln:

$$y - mx = \sqrt{a^2m^2+b^2}$$

$$(y-mx)^2 = a^2m^2 + b^2 \Rightarrow y^2 - 2mxy + m^2x^2 = a^2m^2 + b^2$$

$$m^2(x^2-a^2) - 2mxy + y^2 - b^2 = 0$$

Which is quadratic in 'm'.

here $A = x^2 - a^2$; $B = -2xy$; $C = y^2 - b^2$

$$B^2 - 4AC = 4x^2y^2 - 4(x^2-a^2)(y^2-b^2) = 0$$

$$4x^2y^2 - 4(x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2) = 0$$

$$4x^2y^2 - 4x^2y^2 + 4x^2b^2 + 4a^2y^2 - 4a^2b^2 = 0$$

$$4x^2b^2 + 4a^2y^2 = 4a^2b^2$$

$\div 4a^2b^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

5. Find the envelope of the family of lines $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$; θ being the parameter.

Soln: Given $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \longrightarrow (1)$

Diff. p.w.r.to (1) w.r.t 'θ' we get.

$$-\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta = 0 \longrightarrow (2)$$

Squaring and adding (1) + (2)

$$\left(\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right)^2 + \left(-\frac{x}{a}\sin\theta + \frac{y}{b}\cos\theta\right)^2 = 1^2 + 0^2$$

$$\frac{x^2}{a^2}\cos^2\theta + \frac{y^2}{b^2}\sin^2\theta + \frac{2xy}{ab}\cos\theta\sin\theta$$

$$+ \frac{x^2}{a^2}\sin^2\theta + \frac{y^2}{b^2}\cos^2\theta - \frac{2xy}{ab}\cos\theta\sin\theta = 1$$

$$\frac{x^2}{a^2}\left[\cos^2\theta + \sin^2\theta\right] + \frac{y^2}{b^2}\left[\cos^2\theta + \sin^2\theta\right] = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

6. Find the envelope of $x\sec\theta - y\tan\theta = a$ where 'θ' being the parameter.

Soln:

Given $x\sec\theta - y\tan\theta = a \rightarrow (1)$

$$x\sec\theta = a + y\tan\theta.$$

Squaring both sides.

$$x^2\sec^2\theta = a^2 + 2ay\tan\theta + y^2\tan^2\theta.$$

$$x^2(1+\tan^2\theta) = a^2 + 2ay\tan\theta + y^2\tan^2\theta.$$

$$x^2 + x^2\tan^2\theta = y^2\tan^2\theta + 2ay\tan\theta + a^2$$

$$(y^2 - x^2)\tan^2\theta + 2ay\tan\theta + (a^2 - x^2) = 0$$

(i.e) $(y^2 - x^2)m^2 + 2aym + (a^2 - x^2) = 0.$

Where $m = \tan\theta$ which is a quadratic form in m.

Here $A = y^2 - x^2$, $B = 2ay$; $C = a^2 - x^2$

The envelope is $B^2 - 4AC = 0$.

$$4a^2y^2 - 4(y^2 - x^2)(a^2 - x^2) = 0$$

$$4a^2y^2 - 4[a^2y^2 - x^2y^2 - x^2a^2 + x^4] = 0$$

$$4a^2y^2 - 4a^2y^2 + x^2y^2 \cdot 4 + 4x^2a^2 - 4x^4 = 0$$

$\div$ by $x^2 \Rightarrow$ $y^2 + a^2 - x^2 = 0$

$$x^2 - y^2 = a^2.$$