Topic: 3.2 – RADIUS OF CURVATURE

4.) Find the radius of curvature at any point (x, y) on $y = c \log \sec x/c$.

Solution:

Given $y = c \log \sec x/c$.

$$y_1 = c \cdot \frac{1}{\sec x/c} \cdot \sec \frac{x}{c} \tan \frac{x}{c} \cdot \frac{1}{c}$$

$$y_1 = \tan \frac{x}{c}$$

$$y_1(x,y) = \tan \frac{x}{c}.$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{1}{c} \cdot \sec^2 \frac{x}{c}.$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

$$= \frac{\left[1 + \tan^2 \frac{x}{c}\right]^{3/2}}{\frac{1}{c} \sec^2 \frac{x}{c}} = \frac{\left(\sec^2 \frac{x}{c}\right)^{3/2}}{\frac{1}{c} \sec^2 \frac{x}{c}}.$$

$$\rho = \frac{c \sec^3 \frac{x}{c}}{\sec^2 \frac{x}{c}} = c \sec \frac{x}{c}.$$

$\frac{d}{dx}(\log x) = \frac{1}{x}$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$1 + \tan^2\theta = \sec^2\theta.$

5. find the radius of curvature at $(a,0)$ of the curve $xy^2 = a^3 - x^3$.

solution:

Given $xy^2 = a^3 - x^3$.

$$y^2 = \frac{a^3}{x} - x^2$$

diff. w.r.t 'x'.

$$2y\frac{dy}{dx} = -\frac{a^3}{x^2} - 2x$$

$$y_1 = \frac{-x^3}{2x^2y} - \frac{x}{y}$$

$$y_{1(a,0)} = \infty$$

hence we find $\frac{dx}{dy}$; $xy^2 = a^3 - x^3$

$uv = vdu + udv$

$\frac{u}{v} = \frac{vdu - udv}{v^2}$

$$x\,2y + y^2\frac{dx}{dy} = 0 - 3x^2\frac{dx}{dy}$$

$$2xy + (y^2 + 3x^2)\frac{dx}{dy} = 0.$$

$$\frac{dx}{dy} = \frac{-2xy}{3x^2+y^2}.$$

$$\left(\frac{dx}{dy}\right)_{(a,0)} = 0. \longrightarrow (1)$$

(1) diff w.r.t 'y'

$$\frac{d^2x}{dy^2} = \frac{(3x^2+y^2)\left[-2y\frac{dx}{dy} - 2x\right] - (-2xy)\left[6x\frac{dx}{dy} + 2y\right]}{(3x^2+y^2)^2}.$$

$$\left(\frac{d^2x}{dy^2}\right)_{(a,0)} = \frac{(3a^2+0)\left[(0-2a)-0\right]}{(3a^2+0)^2} = \frac{-6a^3}{9a^4} = \frac{-2}{3a}$$

$$\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}} = \frac{(1+0)^{3/2}}{-\frac{2}{3}a}$$

$$\rho = -\frac{3}{2}a$$

6. find the radius of curvature at the point (c, c) on the curve $xy = c^2$.

solution: Given $xy = c^2$.

$$y = c^2/x$$

$$\frac{dy}{dx} = -c^2/x^2 \quad ; \quad \left(\frac{dy}{dx}\right)_{(c,c)} = -c^2/c^2 = -1$$

$$\frac{d^2y}{dx^2} = \frac{2c^2}{x^3} \qquad \left(\frac{d^2y}{dx^2}\right)_{(c,c)} = 2/c$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+(-1)^2)^{3/2}}{2/c}$$

$$= \frac{c\,2^{3/2}}{2} = \frac{2c\sqrt{2}}{2}$$

$$\rho = c\sqrt{2}.$$

7. find the radius of curvature at any point (x, y) on the curve $y = \frac{1}{2} a\left(e^{x/a} + e^{-x/a}\right)$.

solution:

Radius of curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

Given $y = \frac{a}{2}\left[e^{x/a} + e^{-x/a}\right]$

$= a\left[\frac{e^{x/a} + e^{-x/a}}{2}\right]$

$y = a\cosh\left(\frac{x}{a}\right)$

$y_1 = a\sinh\left(\frac{x}{a}\right)\cdot\frac{1}{a}$.

$y_1 = \sinh\left(\frac{x}{a}\right)$

$y_2 = \cosh\frac{x}{a}\left(\frac{1}{a}\right)$

$\frac{e^{x} - e^{-x}}{2} = \sinh x$

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cosh x) = \sinh x$

$\therefore \sin^2\theta + \cos^2\theta = 1$

$\rho = \frac{\left(1 + \sinh^2\frac{x}{a}\right)^{3/2}}{\frac{1}{a}\cosh\frac{x}{a}} = \frac{\left(\cosh^2\frac{x}{a}\right)^{3/2}}{\frac{1}{a}\cosh\frac{x}{a}}$

$= \frac{a.\cosh^3\frac{x}{a}}{\cosh\frac{x}{a}} = a\cosh^2\frac{x}{a}$

$\rho = a\cosh^2\frac{x}{a}$.

Parametric to cartesian form:

Find the radius of curvature at any point $x = a\cos^3\theta$, $y = a\sin^3\theta$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution: Given $x = a\cos^3\theta$, $y = a\sin^3\theta$.

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\sin\theta\cos^2\theta.$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta = 3a\sin^2\theta\cos\theta.$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}.$$

$$y_1 = \frac{dy}{dx} = -\tan\theta.$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\cdot\frac{d\theta}{dx} = \frac{d}{d\theta}(-\tan\theta)\frac{d\theta}{dx}.$$

$$= \frac{-\sec^2\theta}{-3a\cos^2\theta\sin\theta} = \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\tan^2\theta)^{3/2}}{\frac{1}{3a\cos^4\theta\sin\theta}} = (\sec^2\theta)^{3/2}\,3a\cos^4\theta\sin\theta$$

$$= \sec^3\theta\cdot 3a\cos^4\theta\sin\theta = \frac{1}{\cos^3\theta}3a\cos^4\theta\cdot\sin\theta$$

$$= 3a\sin\theta\cos\theta$$

$$\rho = \frac{3}{2}a\sin 2\theta \qquad [\sin 2A = 2\sin A\cos A].$$