



AN AUTONOMOUS INSTITUTION

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Topic: 2.5 – Integral test

Integral Test:
Cauchy's integral test:
1) $\sum u_n$ is a series of positive terms
and 2) $u_n = f(n)$ be such that
(i) $f(x)$ is continuous in $1 < x < \infty$.
(ii) $f(x)$ decreases as x increases then
the series $\sum u_n$ is convergent or divergent
according as the integral $\int_1^{\infty} f(x) dx$ finite or infinite.

Working Procedure:
1. Find $u_n = f(n)$ [General term]
change 'n' to $x \Rightarrow u_x = f(x)$
2. Ensure that $f'(x) < 0$.
3. Evaluate $\int_1^{\infty} f(x) dx$
4. Conclusion
 $\int_1^{\infty} f(x) dx = \text{finite} \Rightarrow \sum u_n$ is convergent.
 $\int_1^{\infty} f(x) dx = \text{infinite} \Rightarrow \sum u_n$ is divergent.

1. Use integral test to discuss the nature of convergence of the series $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots$

Solu: Given: $\sum u_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots$

Step: To find u_n
Here $u_n = \frac{1}{n(n+1)}$
 $\Rightarrow f(x) = u_x = \frac{1}{x(x+1)} = \frac{1}{x^2+x}$



Step: 2

$$f'(x) = \frac{-(2x+1)}{(x^2+x)^2} < 0$$

Hence, $f(x)$ is decreasing.

Step: 3: Evaluate $\int_1^{\infty} f(x) dx$

$$\begin{aligned}\int_1^{\infty} f(x) dx &= \int_1^{\infty} \frac{1}{x(x+1)} dx \\ &= \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1} \right) dx \\ &= \left[\log x - \log(x+1) \right]_1^{\infty} \\ &= \left[\log \frac{x}{x+1} \right]_1^{\infty} \\ &= \left[\log \frac{1}{1+\frac{1}{x}} \right]_1^{\infty} \\ &= \log 1 - \log \frac{1}{2} \\ &= 0 - \log \frac{1}{2} \\ &= -\log \frac{1}{2} = \text{finite}\end{aligned}$$

Hence, $\sum u_n$ is convergent.



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2. Show that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Solu:

~~Let $f(x) = \frac{1}{x^2+1}$~~

Step:1 To find u_n

Here $u_n = \frac{1}{n^2+1}$

$\Rightarrow f(x) = u_n = \frac{1}{x^2+1}$

Step:2:

$f'(x) = \frac{-2x}{(x^2+1)^2} < 0$

Hence, $f(x)$ is \searrow ing.

Step:3 :

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{dx}{x^2+1}$$
$$= [\tan^{-1}(x)]_1^{\infty}$$
$$= \tan^{-1}(\infty) - \tan^{-1}(1)$$
$$= \frac{\pi}{2} - \frac{\pi}{4}$$
$$= \frac{\pi}{4} \text{ (finite)}$$

$\therefore \int_1^{\infty} f(x) dx$ converges to $\frac{\pi}{4}$

$\therefore \sum u_n$ is convergent.



4. Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n \log n}$

Soln.

$$\text{let } f(x) = \frac{1}{x \log x}, \text{ for } x > 2$$

$$\text{then } \sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \frac{1}{n \log n}$$

$f(x) > 0$ & $f(x)$ is \downarrow in $(2, \infty)$

$$\int_1^{\infty} f(x) dx = \int_2^{\infty} \frac{1}{x \log x} dx$$

$$\text{put } \log x = t \quad \left| \begin{array}{l} x=2, t=\log 2 \\ x=\infty, t=\infty \end{array} \right.$$

$$\frac{1}{x} dx = dt$$

$$= \int_{\log 2}^{\infty} \frac{dt}{t}$$

$$= [\log t]_{\log 2}^{\infty}$$

$$= \log \infty - \log(\log 2)$$

$$= \infty$$

$\therefore \int_2^{\infty} f(x) dx$ is divergent in $(2, \infty)$

By integral test, $\sum_{n=2}^{\infty} f(n)$ is also divergent.

5. Show that the series $e^{-1} + 2e^{-2} + 3e^{-3} + \dots + ne^{-n} + \dots$ converges.

Soln.

$$\text{let } f(x) = x e^{-x}$$

$$\text{then } \sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} n e^{-n}$$

$f(x) > 0$ & is \downarrow in $(1, \infty)$

$$\int_1^{\infty} f(x) dx = \int_1^{\infty} x e^{-x} dx$$

$$= [x e^{-x} - e^{-x}]_1^{\infty}$$

$$= 0 + e^{-1} - 0 + e^{-1} = 2e^{-1}$$

$$= 2/e \text{ (finite)}$$

$\therefore \int_1^{\infty} f(x) dx$ converges. By integral test $\sum_{n=1}^{\infty} f(n)$ also converges.



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6. Test the convergence of the series $\sin \pi + \frac{1}{4} \sin \frac{\pi}{2} + \frac{1}{9} \sin \frac{\pi}{3} + \dots$

Sol:

$$\text{Let } f(x) = \frac{1}{x^2} \sin \frac{\pi}{x}$$
$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin \frac{\pi}{n}$$
$$\int_1^{\infty} f(x) dx = \int_1^{\infty} \frac{1}{x^2} \sin \frac{\pi}{x} dx$$

Put $\frac{\pi}{x} = t$ $x=1, t=\pi$
 $x=\infty, t=0$

$$-\frac{1}{x^2} dx = dt$$
$$= \frac{1}{\pi} \int_0^{\pi} \sin t (-dt)$$
$$= -\frac{1}{\pi} \int_0^{\pi} \sin t dt$$
$$= \frac{1}{\pi} [-\cos t]_0^{\pi}$$
$$= \frac{1}{\pi} [-\cos \pi + \cos 0]$$
$$= \frac{1}{\pi} [4]$$
$$= \frac{4}{\pi} \text{ (finite)}$$

$\therefore \int_1^{\infty} f(x) dx$ converges.

By integral test
 $\sum_{n=1}^{\infty} f(n)$ is also converges.



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