Topic: 2.4 – TEST OF CONVERGENCE

COMPARISON TEST

Test of convergence.

Comparision test for convergence.

If two positive term series $\sum u_n$ & $\sum v_n$ be such that

(i) $\sum v_n$ converges.

(ii) $u_n \le v_n$ for all values of n. then $\sum u_n$ also converges.

(iii) $\sum v_n$ diverges

(iv) $u_n \ge v_n$ for all values of n, then $\sum u_n$ also diverges.

(v) $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite quantity $(\neq 0)$

then $\sum u_n$ & $\sum v_n$ converge or diverge together.

Note

(i) The harmonic series $\sum \frac{1}{n^p}$

(i) Converges if $p > 1$

(ii) diverges if $p \leq 1$

(ii) The Geometric series $\sum r^n$

(i) Converges if $r < 1$

(ii) diverges if $r \geq 1$

(1) Test the convergence of the series

$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots \infty.$$

Soln

$$\text{Gn. } \sum u_n = \frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \cdots + \infty.$$

Step 1

To find n^{th} term (u_n)

In numerator, 1, 3, 5, ... is an A.P

$\therefore t_n = a + (n-1)d.$

$a = 1$

$d = 2$ $\quad t_n = 2n - 1.$

In denominator.

(i) 1, 2, 3 ... are in A.P, $t_n = a + (n-1)d$

$$\boxed{t_n = n}$$

II 2, 3, 4 are in A.P, $t_n = a + (n-1)d$

$$\boxed{t_n = n+1}$$

III 3, 4, 5 are in AP, $\boxed{t_n = n+2}$

$$\therefore u_n = \frac{2n-1}{n(n+1)(n+2)}$$

Step 2

To find v_n

$$v_n = \frac{1}{n^{p-q}}$$

p – higest power of n in denominator
q – hiest power of n in numerator

$$v_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$$

Step 3 To find $\lim_{n\to\infty} \frac{u_n}{v_n}$

$$\frac{u_n}{v_n} = \frac{\left(\frac{2n-1}{n(n+1)(n+2)}\right)}{\left(1/n^2\right)} = \frac{n(2n-1)}{(n+1)(n+2)}$$

$$= \frac{n^2(2 - 1/n)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{2-\frac{1}{n}}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

$$=\frac{2-0}{(1+0)(1+0)}=2$$

Step 4

Conclusion

By comparison test,

$\sum u_n$ & $\sum v_n$ are convergent or divergent together

$\sum v_n = \sum \frac{1}{n^2}$ is of the form

$\sum \frac{1}{n^p}$,

here $p = 2 > 1$

$\therefore \sum v_n$ is convergent

Hence $\sum u_n$ is also convergent.

(4). Test for convergence of the series

$$\frac{1}{4\cdot7\cdot10}+\frac{4}{7\cdot10\cdot13}+\frac{9}{10\cdot13\cdot16}+\cdots$$

Soln. Given $\sum u_n = \frac{1}{4\cdot7\cdot10}+\frac{4}{7\cdot10\cdot13}+\frac{9}{10\cdot13\cdot16}$ (2^2) (3^2)

To find u_n.

In the numerator $t_n = n^2$

In the denominator,

I factors : $4, 7, 10, \ldots$ $t_n = a+(n-1)d$
$= 3n+1$

II factors : $7, 10, 13, \ldots$ $t_n = a+(n-1)d$
$= 3n+4$

III factors : $10, 13, 16, \ldots$ $t_n = 3n+7$

$$\therefore u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)}$$

To find v_n

$$v_n = \frac{1}{n^{p-q}} = \frac{1}{n^{3-2}} = \frac{1}{n}$$

$$\boxed{v_n = \frac{1}{n}}$$

To find $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$

$$\frac{u_n}{v_n} = \frac{n^2}{(3n+1)(3n+4)(3n+7)} \times \frac{n}{1}$$

$$= \frac{n^3}{(3n+1)(3n+4)(3n+7)}$$

$$= \frac{n^3}{n^3} \cdot \frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}$$

$$= \frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(3+\frac{1}{n}\right)\left(3+\frac{4}{n}\right)\left(3+\frac{7}{n}\right)}$$

$$= \frac{1}{27} \text{ . finite.}$$

Conclusion.

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{27} \text{ finite } (\neq 0)$$

Here $\sum v_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$

Here $p = 1$

$\therefore \sum v_n$ = divergent.

$\therefore$ By comparison test, $\sum u_n$ is also divergent

③. Find the nature of the series $\frac{1}{\sqrt{10}}+\frac{1}{\sqrt{20}}+\frac{1}{\sqrt{30}}+\dots$

Soln

Given $\sum u_n = \frac{1}{\sqrt{10}}+\frac{1}{\sqrt{20}}+\frac{1}{\sqrt{30}}+\dots$

To find u_n

In the denominator, 10, 20, 30, ... are in AP

$\therefore t_n = a+(n-1)d = 10n.$

$\therefore u_n = \frac{1}{\sqrt{10n}}$

To find v_n

$$v_n = \frac{1}{n^{p-q}} = \frac{1}{n^{\frac{1}{2}-0}} = \frac{1}{n^{1/2}}$$

To find $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$

$$\frac{u_n}{v_n} = \frac{1}{\sqrt{10n}} \times \frac{n^{1/2}}{1} = \frac{\sqrt{n}}{\sqrt{10}\sqrt{n}} = \frac{1}{\sqrt{10}}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\sqrt{10}}$$

Conclusion

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{\sqrt{10}} \text{ finite } (\neq 0)$$

Here $\sum v_n = \sum \frac{1}{n^{1/2}}$, $p = \frac{1}{2} < 1$

$\therefore \sum v_n$ is divergent

By Comparision test, $\sum u_n$ is also divergent

(14). $\frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \cdots$

Soln Given $\Sigma u_n = \frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \cdots$

To find u_n

$$u_n = \frac{1+2+3+\cdots+n}{1+2^2+3^2+\cdots+n^2} = \frac{\Sigma n}{\Sigma n^2}$$

W.K.T $\Sigma n = \frac{n(n+1)}{2}$ $\Rightarrow$ $\Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$

$$\therefore u_n = \frac{n(n+1)}{2} \times \frac{6}{n(n+1)(2n+1)} = \frac{3}{2n+1}$$

To find v_n

$$v_n = \frac{1}{n^{p-q}} = \frac{1}{n^{3-2}} = \frac{1}{n}$$

To find $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$

$$\frac{u_n}{v_n} = \frac{3}{2n+1} \times \frac{n}{1} = \frac{3n}{2n+1} = \frac{3n}{n(2+1/n)} = \frac{3}{2+1/n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{3}{2+\frac{1}{n}} = \frac{3}{2} = \text{finite } (\neq 0)$$

Conclusion

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{3}{2} \text{ finite } (\neq 0)$$

Here $\Sigma v_n = \Sigma \frac{1}{n}$, $p = 1$

$\Rightarrow \Sigma v_n$ is divergent

By comparision Test, Σu_n is also divergent.

Discuss the convergence or divergence of the series. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \dots + \infty$

Soln

To find u_n

num: $2, 3, 4, \dots n^p$ $t_n = n+1$

Den: $1, 2, 3, \dots n^p$ $t_n = n$

$u_n = \frac{n+1}{n^p}$

To find v_n $v_n = \frac{1}{n^{p-q}} = \frac{1}{n^{p-1}}$

$\frac{u_n}{v_n} = \frac{n+1}{n^p} \times \frac{n^{p-1}}{1} = \frac{n+1}{n} = \frac{n(1+1/n)}{n} = 1 + \frac{1}{n}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$. finite $(\neq 0)$

Therefore, by comparision test, Σu_n & Σv_n are convergent or divergent together

bt, $v_n = \Sigma \frac{1}{n^{p-1}}$ is of the form $\Sigma \frac{1}{n^p}$

Σv_n is convergent if $(p-1) > 1$

$\Rightarrow \Sigma u_n$ is also convergent if $(p-1) > 1$
i.e., $p > 2$

Σv_n is divergent if $(p-1) < 1$

$\Rightarrow \Sigma u_n$ is also divergent if $(p-1) \leq 1$ i.e.,
$p \leq 2$

⑤. $\frac{1}{\sqrt{1\cdot 2}} + \frac{1}{\sqrt{2\cdot 3}} + \frac{1}{\sqrt{3\cdot 4}} + \cdots$

Soln $u_n = \frac{1}{\sqrt{n(n+1)}}$ $\quad v_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ Divergent.

⑥. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \cdots$

$u_n = \frac{n^2}{n!} = \frac{n^2}{n(n-1)(n-2)!} = \frac{n}{(n-1)(n-2)!}$

$v_n = \frac{1}{(n-2)!}$ convergent.

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$

⑦. $\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \cdots$

$u_n = \frac{\sqrt{n+1}-1}{(n+2)^3-1}$ $\quad v_n = \frac{1}{n^{5/2}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ Convergent.

⑧. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \cdots$

$u_n = \frac{n^n}{(n+1)^{n+1}}$ $\quad v_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{e}$

divergent. $\left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$