Topic: 1.5 –PROPERTIES IF EIGEN VALUES & EIGEN VECTORS

Properties of Eigen values and Eigen vectors

* The sum of the Eigen values of a matrix is the sum of the elements of the principal (or) diagonal.
* The product of the Eigen values of a matrix is the determinant value of the given matrix say $|A|$.
* A square matrix A and its transpose A^T have the same Eigen values.
* The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.
* If λ is an Eigen value of a matrix A, then $1/\lambda$ is the Eigen value of the matrix A^{-1}.
* If λ is an Eigen value of an orthogonal matrix, then $1/\lambda$ is also one of its Eigen values.
* The Eigen value of a real symmetric matrix are real numbers.
* The similar matrices have the same Eigen value.
* Two Eigen vectors X_1 and X_2 are called orthogonal if $X_1^T X_2 = 0$
* If A and B are two matrices and B is non-singular, then A and $B^{-1}AB$ have the same Eigen values.

Problems:

1. Find the sum and product of Eigen values of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & 2 & 0 \end{bmatrix}$

Gn $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & 2 & 0 \end{bmatrix}$

i) Sum of Eigen values = Sum of diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = -2 + 1 + 0$$

$$\boxed{\lambda_1 + \lambda_2 + \lambda_3 = -1}$$

ii) Product of Eigen values = $|A|$

$$\lambda_1 \lambda_2 \lambda_3 = -2(12) - 2(-6) - 3(5)$$

$$\boxed{\lambda_1 \lambda_2 \lambda_3 = -27}$$

2. The product of two Eigen values of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third Eigen value.

Gn. the product of two Eigen values is 16

To find the third Eigen value λ_3 by using property

Product of Eigen values = $|A|$

$$\lambda_1 \lambda_2 \lambda_3 = 6(9-1) + 2(-6+2) + 2(2-6)$$

$$16 . \lambda_3 = 6(8) + 2(-4) + 2(-4)$$

$$16 . \lambda_3 = 48 - 8 - 8$$

$$\lambda_3 = 32/16$$

$$\boxed{\lambda_3 = 2}$$

3. If 3 and 15 are two Eigen values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$. Find $|A|$ without expanding the determinant.

Gn. $\lambda_1 = 3$; $\lambda_2 = 15$

To find $|A|$ without expanding the determinant by using property

$$\lambda_1 + \lambda_2 + \lambda_3 = d_1 + d_2 + d_3$$

$$3 + 15 + \lambda_3 = 8 + 7 + 3$$

$$\boxed{\lambda_3 = 0}$$

W.k.t. Product of Eigen values = $|A|$

$$\lambda_1 \lambda_2 \lambda_3 = |A|$$

$$3 . 15 . 0 = |A|$$

$$\boxed{|A| = 0}$$

4. If 2, 2, 3 are the Eigen values of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, find the Eigen values of A^T.

By property, A square matrix A and its transpose A^T have the same Eigen values.

Gn Eigen values of $A = 2, 2, 3$

Eigen values of $A^T = 2, 2, 3$.

5. Find the Eigen values of adjoint of A if $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \frac{\text{adj}.A}{|A|}$$

$$A^{-1}|A| = \text{adj}.A$$

Eigen values of $\text{adj}.A = |A|$. Eigen values of A^{-1} —— ①

$$|A| = 3(4-0) - 2(0-0) + 1(0)$$

$$|A| = 12$$

Eigen values of A is 1, 3, 4 [because Eigen values of triangular matrix is the coefficients on the diagonals]

Eigen values of A^{-1} is $1, 1/3, 1/4$.

Sub all in ① we get, Eigen values of adj.A $= 12\left(1, \frac{1}{3}, \frac{1}{4}\right)$

$= 12, 4, 3$

Orthogonal Matrices :

A square matrix A is said to be orthogonal if

$$AA^T = A^TA = I \quad (\because AA^{-1} = A^{-1}A = I)$$

A matrix A is orthogonal if $A^T = A^{-1}$

1. Check whether the matrix B is orthogonal $B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ \theta & 0 & 1 \end{bmatrix}$

Given $B = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$B^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BB^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta + 0 & -\sin\theta\cancel{\cos\theta} + \cos\cancel{\theta\sin\theta} + 0 & 0+0+0 \\ -\sin\theta\cancel{\cos\theta} + \sin\theta\cancel{\cos\theta} + 0 & \sin^2\theta + \cos^2\theta + 0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

Similary $B^T B = I$

Hence the given matrix is orthogonal.