Topic: 2. 7– D'ALEMBERT'S RATIO TEST

5. Test the convergent of the series.

$$\sum_{n=1}^{\infty} \frac{3.6.9\cdots 3n}{4.7.10\cdots(3n+1)} \cdot \frac{5^n}{3n+2}.$$

Soln

$$u_n = \frac{3.6.9\cdots 3n}{4.7.10\cdots(3n+1)} \cdot \frac{5^n}{3n+2}.$$

$$u_{n+1} = \frac{3.6.9\cdots 3(n+1)}{4.7.10\cdots 3(n+1+1)} \cdot \frac{5^{n+1}}{(3(n+1)+2)}$$

$$= \frac{3.6.9\cdots 3n(3n+3)}{4.7.10\cdots(3n+1)(3n+4)} \cdot \frac{5^{n+1}}{3n+5}$$

$$\frac{u_{n+1}}{u_n} = \frac{(3n+3)(3n+2)}{(3n+4)(3n+5)} \cdot 5$$

$$= \frac{n^2\left(n+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)}{n^2\left(3+4/n\right)\left(3+5/n\right)} \cdot 5$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} 5\,\frac{\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)}{\left(3+\frac{4}{n}\right)\left(3+5/n\right)}$$

$$= \frac{5\ 3.3}{3.3} = 5 > 1$$

Hence, by ratio test $\sum u_n$ is divergent

(5). Test the convergence of the series

$$1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\cdots$$

Soln $\quad u_n=\frac{n^p}{n!} \qquad u_{n+1}=\frac{(n+1)^p}{(n+1)!}$

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)^p}{(n+1)!}\times\frac{n!}{n^p}=\frac{(n+1)^p}{(n+1)n!}\cdot\frac{n!}{n^p}$$

$$=\frac{(n+1)^p}{n+1}\cdot\frac{1}{n^p}=\frac{n^p(1+1/n)^p}{n+1}\cdot\frac{1}{n^p}$$

$$=\frac{\left(1+\frac{1}{n}\right)^p}{n+1}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{(1+1/n)^p}{n+1}=0<1$$

Hence by ratio test, Σu_n is convergent

(4). Test the convergence of the series

$$x+2x^2+3x^3+4x^4+\cdots$$

Soln $\quad u_n=nx^n \;;\; u_{n+1}=(n+1)x^{n+1}$

$$\frac{u_{n+1}}{u_n}=\frac{(n+1)x^n\cdot x}{n\,x^n}=\frac{(n+1)x}{n}=\left(1+\frac{1}{n}\right)x$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)x=x$$

If $x < 1$ then $\sum u_n$ converges.

If $x > 1$ then $\sum u_n$ diverges.

If $x = 1$ then $\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = 1$ but fails

$\therefore S_n = x + 2x^2 + 3x^3 \ldots\ldots n$ terms.

$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \sum u_n, \lim_{n \to \infty} n = \infty.$

$\therefore \sum u_n$ is divergent

$\sum u_n$ is convergent if $x < 1$ &

divergent if $x \geq 1$