



Topic: 2. 6 – D'ALEMBERT'S RATIO TEST

D'Alembert's ratio test.

The series $\sum u_n$ of positive terms is

convergent if $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} < 1$

is divergent if $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} > 1$

if $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 1$, the test fails.

NOTE

Usually we try D'Alembert's ratio test for the following cases.

- (i) u_n is not of the order $\frac{1}{n^k}$
- (ii) u_n involves variables like x^n, x^{2^n}, \dots
- (iii) u_n involves $n!, (n!), (n!)^2, \dots$
- (iv) The number of factors in the N or D, increase steadily from term to term like

$$\frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 5} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} + \dots$$



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Q. Discuss the convergence of the series

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \dots$$

Soln

$$u_n = \frac{n^2}{n!} \quad (\text{is not of the order } \frac{1}{n^k})$$
$$u_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

To find $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n}$

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{(n+1)^2}{(n+1)!} \times \frac{n!}{n^2} \\ &= \frac{(n+1)^2}{(n+1)!} \times \frac{n!}{n^2} = \frac{(n+1)^2}{n!(n+1)} \times \frac{n!}{n^2} \\ &= \frac{(n+1)}{n^2} = \frac{n}{n^2} + \frac{1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \end{aligned}$$
$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n^2} \right) = 0.$$

Conclusion

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = 0 < 1$$

\therefore By ratio test $\sum u_n$ is convergent.



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③. Test the convergence of the series

$$\frac{3}{4+1} + \frac{3^2}{4^2+1} + \frac{3^3}{4^3+1} + \dots$$

Soln

$$u_n = \frac{3^n}{4^n + 1} \quad \left[\text{This is not of the order } \frac{1}{n^n} \right]$$
$$u_{n+1} = \frac{3^{n+1}}{4^{n+1} + 1}$$
$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}}{4^{n+1} + 1} \times \frac{4^n + 1}{3^n}$$
$$= \frac{3^n \cdot 3}{4^{n+1} \left(1 + \frac{1}{4^{n+1}}\right)} \cdot \frac{4^n \left(1 + \frac{1}{4^n}\right)}{3^n}$$
$$= \frac{3}{4} \left(\frac{1 + \frac{1}{4^n}}{1 + \frac{1}{4^{n+1}}} \right)$$
$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \frac{3}{4} < 1$$

\therefore Hence by Ratio test, $\sum u_n$ is convergent.



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③ Test the convergence of the series.

$$\frac{4}{3} + \frac{4 \cdot 7}{3 \cdot 5} + \frac{4 \cdot 7 \cdot 10}{3 \cdot 5 \cdot 7} + \dots$$

Sol

$4, 7, 10, \dots$ are in AP, $a=4$
 $d=3$

$$t_n = a + (n-1)d$$
$$= 3n+1$$

$3, 5, 7, \dots$ are in AP, $a=3$
 $d=2$

$$t_n = 2n+1$$
$$\therefore U_n = \frac{4 \cdot 7 \cdot 10 \dots (3n+1)}{3 \cdot 5 \cdot 7 \dots (2n+1)}$$
$$U_{n+1} = \frac{4 \cdot 7 \cdot 10 \dots (3(n+1)+1)}{3 \cdot 5 \cdot 7 \dots (2(n+1)+1)}$$
$$= \frac{4 \cdot 7 \cdot 10 \dots (3n+1)(3n+4)}{3 \cdot 5 \cdot 7 \dots (2n+1)(2n+3)}$$
$$\frac{U_{n+1}}{U_n} = \frac{3n+4}{2n+3} = \frac{n(3+\frac{4}{n})}{n(2+\frac{3}{n})}$$
$$\lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = \lim_{n \rightarrow \infty} \frac{3+\frac{4}{n}}{2+\frac{3}{n}} = \frac{3}{2} > 1$$

\therefore Hence by ratio test $\sum U_n$ is divergent.



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4) Test the convergent of the series

$$\frac{1^2}{4^2} + \frac{1^2 \cdot 5^2}{4^2 \cdot 8^2} + \frac{1^2 \cdot 5^2 \cdot 9^2}{4^2 \cdot 8^2 \cdot 12^2} + \dots$$

Sol.

$\Rightarrow 1, 5, 9$ are in A.P. $t_n = a + (n-1)d$
 $a=1, d=4$ $t_n = 4n-3$

$\Rightarrow 4, 8, 12$ are in A.P. $t_n = 4n$

$$U_n = \frac{1^2 \cdot 5^2 \cdot 9^2 \dots (4n-3)^2}{4^2 \cdot 8^2 \dots (4n)^2}$$
$$U_{n+1} = \frac{1^2 \cdot 5^2 \dots (4n-3)^2 (4n+1)^2}{4^2 \cdot 8^2 \dots (4n)^2 (4n+4)^2}$$
$$\frac{U_{n+1}}{U_n} = \frac{(4n+1)^2}{(4n+4)^2} = \frac{n^2 \left(4 + \frac{1}{n}\right)^2}{n^2 \left(4 + \frac{4}{n}\right)^2} = \frac{\left(4 + \frac{1}{n}\right)^2}{\left(4 + \frac{4}{n}\right)^2}$$
$$\lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = \lim_{n \rightarrow \infty} \frac{\left(4 + \frac{1}{n}\right)^2}{\left(4 + \frac{4}{n}\right)^2} = \frac{16}{16} = 1$$

$\lim_{n \rightarrow \infty} \frac{U_{n+1}}{U_n} = 1 \Rightarrow$ Ratio test fails.

Apply Raabe's Test.

$$\frac{U_n}{U_{n+1}} - 1 = \frac{(4n+4)^2 - (4n+1)^2}{(4n+1)^2} = \frac{24n+15}{(4n+1)^2}$$
$$\lim_{n \rightarrow \infty} n \left(\frac{U_n}{U_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} \frac{n \cdot n \left(24 + \frac{15}{n}\right)}{n^2 \left(4 + \frac{1}{n}\right)^2} = \frac{3}{2} > 1$$

$\therefore \sum U_n$ is convergent



SNS COLLEGE OF ENGINEERING
Kurumbapalayam (Po), Coimbatore – 641 107

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