



Topic: 2.4 – TEST OF CONVERGENCE

COMPARISON TEST

Test of convergence.

Comparison test for convergence.

If two positive term series $\sum u_n$ & $\sum v_n$ be such that

- (i) $\sum v_n$ converges.
- (ii) $u_n \leq v_n$ for all values of n , then $\sum u_n$ also converges.
- (iii) $\sum v_n$ diverges.
- (iv) $u_n > v_n$ for all values of n , then $\sum u_n$ also diverges.
- (v) $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \text{finite quantity } (\neq 0)$

then $\sum u_n$ & $\sum v_n$ converge or diverge together.



Note

(i) The harmonic series $\sum \frac{1}{n^p}$

(i) Converges if $p > 1$

(ii) Diverges if $p \leq 1$

(ii) The Geometric series $\sum r^n$

(i) Converges if $r < 1$

(ii) Diverges if $r \geq 1$

Q. Test the convergence of the series

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \frac{5}{3 \cdot 4 \cdot 5} + \dots \infty$$

Soln

$$\text{Gn. } \sum u_n = \frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 3 \cdot 4} + \dots \infty$$

Step 1

To find n^{th} term (u_n)

In numerator, 1, 3, 5, ... is an A.P.

$$\therefore t_n = a + (n-1)d$$

$$a = 1$$

$$d = 2$$

$$t_n = 2n - 1$$



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In denominator,

Ex 1 $1, 2, 3, \dots$ are in A.P., $t_n = a + (n-1)d$
 $t_n = n$

Ex 2 $2, 3, 4, \dots$ are in A.P., $t_n = a + (n-1)d$
 $t_n = n+1$

Ex 3 $3, 4, 5, \dots$ are in A.P., $t_n = n+2$

$\therefore u_n = \frac{2n-1}{n(n+1)(n+2)}$

Step 2
To find v_n
 $v_n = \frac{1}{n^p \cdot q}$ $p = \text{highest power of } n \text{ in denominator}$
 $q = \text{highest power of } n \text{ in numerator}$
 $v_n = \frac{1}{n^{3-1}} = \frac{1}{n^2}$

Step 3 To find $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$

$$\frac{u_n}{v_n} = \frac{\left(\frac{2n-1}{n(n+1)(n+2)} \right)}{\left(\frac{1}{n^2} \right)} = \frac{n(2n-1)}{(n+1)(n+2)}$$
$$= \frac{n^2 \left(2 - \frac{1}{n} \right)}{n^2 \left(1 + \frac{1}{n} \right) \left(1 + \frac{2}{n} \right)}$$



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$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{2 - \frac{1}{n}}{\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)}$$
$$= \frac{2 - 0}{(1+0)(1+0)} = 2$$

Steps

conclusion

by comparison test,

$\sum u_n$ & $\sum v_n$ are convergent or
divergent together

$\sum v_n = \sum \frac{1}{n^2}$ is of the form

$\sum \frac{1}{n^p}$

Here $p = 2 > 1$

$\therefore \sum v_n$ is convergent

Hence $\sum u_n$ is also convergent.



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Q. Test for convergence of the series
 $\frac{1}{4 \cdot 7 \cdot 10} + \frac{4}{7 \cdot 10 \cdot 13} + \frac{9}{10 \cdot 13 \cdot 16} + \dots$

Sol. Given $\sum u_n = \frac{1}{4 \cdot 7 \cdot 10} + \frac{4}{7 \cdot 10 \cdot 13} + \frac{9}{10 \cdot 13 \cdot 16}$

To find u_n .

In the numerator $t_n = n^2$

In the denominator,

I factors: 4, 7, 10, ... $t_n = a + (n-1)d$
 $= 3n+1$

II factors: 7, 10, 13, ... $t_n = a + (n-1)d$
 $= 3n+4$

III factors: 10, 13, 16, ... $t_n = 3n+7$

$$\therefore u_n = \frac{n^2}{(3n+1)(3n+4)(3n+7)}$$

To find v_n

$$v_n = \frac{1}{n^p - q} = \frac{1}{n^3 - 9} = \frac{1}{n}$$

$$v_n = \frac{1}{n}$$



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To find $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$

$$\frac{u_n}{v_n} = \frac{n^2}{(2n+1)(3n+2)(2n+7)} \times \frac{n}{1}$$
$$= \frac{n^3}{(2n+1)(3n+2)(2n+7)}$$
$$= \frac{n^3}{n^3} \cdot \frac{1}{\left(2+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(2+\frac{7}{n}\right)}$$
$$= \frac{1}{\left(2+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(2+\frac{7}{n}\right)}$$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{1}{\left(2+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)\left(2+\frac{7}{n}\right)}$

$$= \frac{1}{27} = \text{finite.}$$

Conclusion.

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{27} \text{ finite } (\neq 0)$$

Here $\sum v_n = \sum \frac{1}{n}$ is of the form $\sum \frac{1}{n^p}$

Here $p=1$

$\therefore \sum v_n = \text{divergent.}$

\therefore By comparison test, $\sum u_n$ is also divergent.



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③. Find the nature of the series $\frac{1}{\sqrt{10}} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{50}} + \dots$

Soln Given $\sum u_n = \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{50}} + \dots$

To find u_n
In the denominator, 10, 30, 50, ... are in AP
 $\therefore t_n = a + (n-1)d = 10n$
 $\therefore u_n = \frac{1}{\sqrt{10n}}$

To find v_n
 $v_n = \frac{1}{n^{p-q}} = \frac{1}{n^{\frac{1}{2}-0}} = \frac{1}{n^{1/2}}$

To find $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$
 $\frac{u_n}{v_n} = \frac{1}{\sqrt{10n}} \times \frac{n^{1/2}}{1} = \frac{\sqrt{n}}{\sqrt{10}\sqrt{n}} = \frac{1}{\sqrt{10}}$
 $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{\sqrt{10}}$

Conclusion
 $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{\sqrt{10}}$ finite ($\neq 0$)
Here $\sum v_n = \sum \frac{1}{n^{1/2}}$, $p = \frac{1}{2} < 1$
 $\therefore \sum v_n$ is divergent
By comparison test, $\sum u_n$ is also divergent



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④. $\frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \dots$

Soln $\sum u_n = \frac{1}{1^2} + \frac{1+2}{1^2+2^2} + \frac{1+2+3}{1^2+2^2+3^2} + \dots$

To find u_n
 $u_n = \frac{1+2+3+\dots+n}{1^2+2^2+3^2+\dots+n^2} = \frac{\frac{n(n+1)}{2}}{\frac{n(n+1)(2n+1)}{6}}$

$\therefore u_n = \frac{n(n+1)}{2} \times \frac{6}{n(n+1)(2n+1)} = \frac{3}{2n+1}$

To find v_n
 $v_n = \frac{1}{n^p} = \frac{1}{n^{2-1}} = \frac{1}{n}$

To find $\lim_{n \rightarrow \infty} \frac{u_n}{v_n}$
 $\frac{u_n}{v_n} = \frac{3}{2n+1} \times \frac{n}{1} = \frac{3n}{2n+1} = \frac{3n}{n(2+\frac{1}{n})} = \frac{3}{2+\frac{1}{n}}$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} \frac{3}{2+\frac{1}{n}} = \frac{3}{2} = \text{finite } (\neq 0)$

Conclusion $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{3}{2}$ finite ($\neq 0$)

Here $\sum v_n = \sum \frac{1}{n}$, $p=1$
 $\Rightarrow \sum v_n$ is divergent

By comparison Test, $\sum u_n$ is also divergent.



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Discuss the convergence or divergence of the series. $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \dots + \infty$

Soln.

To find u_n
num: 2, 3, 4, ... AP $t_n = n+1$
Den: 1, 2, 3, ... AP $t_n = n$
 $u_n = \frac{n+1}{n^p}$

To find v_n
 $v_n = \frac{1}{n^{p-1}} = \frac{1}{n^{p-1}}$

$$\frac{u_n}{v_n} = \frac{n+1}{n^p} \times \frac{n^{p-1}}{1} = \frac{n+1}{n} = \frac{n(1+\frac{1}{n})}{n} = 1 + \frac{1}{n}$$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{n \rightarrow \infty} (1 + \frac{1}{n}) = 1$ - finite ($\neq 0$)

Therefore, by comparison test, $\sum u_n$ & $\sum v_n$ are convergent or divergent together

bt, $v_n = \sum \frac{1}{n^{p-1}}$ is of the form $\sum \frac{1}{n^p}$

$\sum v_n$ is convergent if $(p-1) > 1$
 $\Rightarrow \sum u_n$ is also convergent if $(p-1) > 1$
i.e. $p > 2$

$\sum v_n$ is divergent if $(p-1) < 1$
 $\Rightarrow \sum u_n$ is also divergent if $(p-1) < 1$
i.e. $p < 2$



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⑤. $\frac{1}{\sqrt{1 \cdot 2}} + \frac{1}{\sqrt{2 \cdot 3}} + \frac{1}{\sqrt{3 \cdot 4}} + \dots$

Ratio $u_n = \frac{1}{\sqrt{n(n+1)}} \quad v_n = \frac{1}{n}$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$ Divergent.

⑥. $1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \dots$

$u_n = \frac{n^2}{n!} = \frac{n^2}{n(n-1)(n-2)!} = \frac{n}{(n-1)(n-2)!}$

$v_n = \frac{1}{(n-2)!}$ convergent.

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$

⑦. $\frac{\sqrt{2}-1}{3^2-1} + \frac{\sqrt{3}-1}{4^2-1} + \frac{\sqrt{4}-1}{5^2-1} + \dots$

$u_n = \frac{\sqrt{n+1}-1}{(n+2)^2-1} \quad v_n = \frac{1}{n^{5/2}}$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$ Convergent.

⑧. $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \frac{4^4}{5^5} + \dots$

$u_n = \frac{n^n}{(n+1)^{n+1}} \quad v_n = \frac{1}{n} \quad \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{e}$

Divergent. $\left[\because \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \right]$