Topic: 2. 5– D'ALEMBERT'S RATIO TEST

D' Alembert's ratio test.

The series $\sum u_n$ of positive terms is convergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$

is divergent if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$

If $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, the test fails.

Note

Usually we try D' Alembert's ratio test for the following cases.

(i) u_n is not of the order $\frac{1}{n^p}$

(ii) u_n involves variables like $x^n, x^{2n}, \ldots$

(iii) u_n involves $n!, (n+1), (n!)^2, \ldots$

(iv) The number of factors in the Nr & Dr. increase steadily from term to term like

$\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \cdots$

2. Discuss the convergence of the series

$$1 + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{4^2}{4!} + \cdots$$

Soln

$$u_n = \frac{n^2}{n!} \quad \text{(is not of the order } \frac{1}{n^p}\text{)}$$

$$u_{n+1} = \frac{(n+1)^2}{(n+1)!}$$

To find $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$.

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(n+1)!} \times \frac{n!}{n^2}$$

$$= \frac{(n+1)^2}{(n+1)!} \times \frac{n!}{n^2} = \frac{(n+1)^2}{\cancel{n!}(n+1)} \cdot \frac{\cancel{n!}}{n^2}$$

$$= \frac{(n+1)}{n^2} = \frac{n}{n^2} + \frac{1}{n^2} = \frac{1}{n} + \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{1}{n} + \frac{1}{n^2}\right) = 0.$$

Conclusion

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 0 < 1$$

$\therefore$ By ratio test $\sum u_n$ is convergent.

(2). Test the convergence of the series

$$\frac{3}{4+1} + \frac{3^2}{4^2+1} + \frac{3^3}{4^3+1} + \cdots\cdots$$

soln

$$u_n = \frac{3^n}{4^n+1}$$ [u_n is not of the order $\frac{1}{n^p}$]

$$u_{n+1} = \frac{3^{n+1}}{4^{n+1}+1}$$

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}}{4^{n+1}+1} \times \frac{4^n+1}{3^n}$$

$$= \frac{3^n . 3}{4^{n+1}\left(1+\frac{1}{4^{n+1}}\right)} \cdot \frac{4^n\left(1+\frac{1}{4^n}\right)}{3^n}$$

$$= \frac{3}{4}\left(\frac{1+\frac{1}{4^n}}{1+\frac{1}{4^{n+1}}}\right)$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{3}{4} < 1$$

∴ Hence by ratio test, Σu_n is convergent.

(3) Test the convergence of the series.

$$\frac{4}{3} + \frac{4\cdot 7}{3\cdot 5} + \frac{4\cdot 7\cdot 10}{3\cdot 5\cdot 7} + \cdots$$

Soln: $4, 7, 10, \ldots$ are in AP. $a = 4$, $d = 3$

$$t_n = a + (n-1)d$$
$$= 3n+1$$

$3, 5, 7, \ldots$ are in AP. $a = 3$, $d = 2$.

$$t_n = 2n+1$$

$$\therefore u_n = \frac{4\cdot 7\cdot 10 \cdots (3n+1)}{3\cdot 5\cdot 7 \cdots (2n+1)}$$

$$u_{n+1} = \frac{4\cdot 7\cdot 10 \cdots (3(n+1)+1)}{3\cdot 5\cdot 7 \cdots (2(n+1)+1)}$$
$$= \frac{4\cdot 7\cdot 10 \cdots (3n+1)(3n+4)}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)}$$

$$\frac{u_{n+1}}{u_n} = \frac{3n+4}{2n+3} = \frac{n(3+4/n)}{n(2+3/n)} =$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{3+\frac{4}{n}}{2+3/n} = 3/2 > 1$$

$\therefore$ Hence by ratio test $\sum u_n$ is divergent.

(4). Test the convergent of the series

$$\frac{1^2}{4^2}+\frac{1^2.5^2}{4^2.8^2}+\frac{1^2.5^2.9^2}{4^2.8^2.12^2}+\cdots$$

Soln.

1, 5, 9 are in A.P $\quad t_n = a+(n-1)d$

$a=1, d=4 \quad t_n = 4n-3.$

4, 8, 12 are in A.P. $\quad t_n = 4n.$

$$u_n = \frac{1^2.5^2.9^2\cdots(4n-3)^2}{4^2.8^2\cdots\cdots(4n)^2}$$

$$u_{n+1} = \frac{1^2.5^2\cdots\cdots(4n-3)^2(4n+1)^2}{4^2.8^2\cdots\cdots(4n)^2(4n+4)^2}$$

$$\frac{u_{n+1}}{u_n} = \frac{(4n+1)^2}{(4n+4)^2} = \frac{n^2\left(4+\frac{1}{n}\right)^2}{n^2\left(4+\frac{4}{n}\right)^2} = \frac{\left(4+\frac{1}{n}\right)^2}{\left(4+\frac{4}{n}\right)^2}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{\left(4+\frac{1}{n}\right)^2}{\left(4+\frac{4}{n}\right)^2} = \frac{16}{16} = 1$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = 1 \Rightarrow \text{Ratio test fails.}$$

Apply Raabe's Test.

$$\frac{u_n}{u_{n+1}} - 1 = \frac{(4n+4)^2-(4n+1)^2}{(4n+1)^2} = \frac{24n+15}{(4n+1)^2}.$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\frac{n.n\left(24+\frac{15}{n}\right)}{n^2\left(4+\frac{1}{n}\right)^2} = \frac{3}{2} > 1$$

$\therefore \Sigma u_n$ is convergent