Buckinghams π-theorem

It states, "If there are n variables (independent and dependent variables) in a physical phenomenon and if these variables contains m fundamental dimensions (M, L, T), then the variables are arranged into $(n-m)$ dimensionless terms. Each term is called π-term".

Let X_1 = dependent variable

$X_2, X_3 \ldots X_n$ = independent variables, then

$$\boxed{X_1 = f(X_2, X_3 \ldots X_n)}$$

Can also be written as

$$\boxed{f(X_1, X_2, X_3, \ldots X_n) = 0.}$$

According to Buckingham's π theorem it can be written as

$$\boxed{f_1(\pi_1, \pi_2, \ldots \pi_{n-m}) = 0} \quad (1)$$

∴ Each π term contains $m+1$ variables, where m is the number of fundamental dimensions and also called as repeating variables.

$$\left.\begin{aligned} \pi_1 &= X_2^{a_1} \cdot X_3^{b_1} \cdot X_4^{c_1} \cdot X_1 \\ \pi_2 &= X_2^{a_2} \cdot X_3^{b_2} \cdot X_4^{c_2} \cdot X_5 \\ \pi_{n-m} &= X_2^{a_{n-m}} X_3^{b_{n-m}} \cdot X_4^{c_{n-m}} \cdot X_n \end{aligned}\right\} \text{Equations}$$

Each equation is solved by the principle of dimensional homogenity and values of a_1, b_1, c_1, etc are obtained. Then these values are substituted in equation and values of $\pi_1, \pi_2, \pi_3 \ldots \pi_{n-m}$ are obtained. These values of π's are then substituted in equation (1). Final equation is as follows

$$\pi_1 = \phi[\pi_2, \pi_3, \ldots \pi_{n-m}]$$

$$\pi_2 = \phi_1[\pi_1, \pi_3, \ldots \pi_{n-m}]$$

Method of selecting repeating variables:
(Rules)
1) No. of repeating variables are equal to the no of fundamental dimensions of the problem.
2) Dependent variable should not be selected.
3) It should be selected in such a way that one variable contains geometric property, other variable contains flow property and third variable contains fluid property.

Geometric property:
(i) Length (ii) diameter. (iii) height, etc

Flow property:
(i) velocity (ii) acceleration, (iii) discharge

Fluid property:
(i) viscosity (ii) density (iii) specific weight

4) should not form a dimensionless group (should contains all three fundamental dimension)
5) No two repeating variables should have same dimensions.

Examples: (i) d, v, ρ (iii) l, v, ρ
(ii) d, v, μ (iv) l, v, μ.

The resisting force R depends upon the length of the aircraft l, velocity v, air viscosity, μ, air density ρ and bulk modulus of air K. Express the functional relationship between these variables and the resisting force

Solution: (68)

$$R = f(l, v, \mu, \rho, K)$$

$$f(R, l, v, \mu, \rho, K) = 0$$ Step 1

Total no. of variables = 6 (n)
no. of fundamental dimensions = 3 (m)

No. of dimensionless π-terms $= (n-m) = 6-3 = 3$. Hence

$$f_1(\pi_1, \pi_2, \pi_3) = 0.$$

Each π-term = $m+1$

where m = repeating variables.

R, l, V, μ, ρ and K. — total variable

Step 2

l, V, ρ — repeating variables.

Now it can be written as.

$$\left.\begin{aligned} \pi_1 &= l^{a_1} . V^{b_1} . \rho^{c_1} . R \\ \pi_2 &= l^{a_2} . V^{b_2} . \rho^{c_2} . \mu \\ \pi_3 &= l^{a_3} . V^{b_3} . \rho^{c_3} . K \end{aligned}\right\}$$

Step 3

Each π-term is solved by the principle of homogenity

$\pi_1 = M^0L^0T^0 = L^{a_1} . (LT^{-1})^{b_1} . (ML^{-3})^{c_1} . M.LT^{-2}$

Power of $M = 0 = c_1 + 1$ — ① $\therefore c_1 = -1$

Power of $L = 0 = a_1 + b_1 - 3c_1 + 1$ — ②

Power of $T = 0 = -b_1 - 2$ — ③ $\therefore b_1 = -2$

Substituting in ②

$a_1 - 2 - 3(-1) + 1 = 0$

$a_1 - 2 + 3 + 1 = 0$

Step 4

$a_1 + 2 = 0$

$a_1 = -2$

Now in $\pi_1 = l^{-2} . V^{-2} . \rho^{-1} . R$.

$$\boxed{\pi_1 = \frac{R}{l^2V^2\rho} = \frac{R}{\rho l^2 V^2}}$$

Now π_2,

$\pi_2 = l^{a_2} . V^{b_2} . \rho^{c_2} . \mu$

$\pi_2 = M^0L^0T^0 = L^{a_2} . (LT^{-1})^{b_2} . (ML^{-3})^{c_2} . ML^{-1}T^{-1}$

Power of $M = 0 = c_2 + 1 \rightarrow$ ① $\therefore \boxed{c_2 = -1}$

" " $L = 0 = a_2 + b_2 - 3c_2 - 1 \rightarrow$ ②

" " $T = 0 = -b_2 - 1 \rightarrow$ ③ $\therefore \boxed{b_2 = -1}$

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$$a_2 + b_2 - 3c_2 - 1 = 0$$

$$a_2 - 1 - 3(-1) - 1 = 0$$

$$a_2 - 1 + 3 - 1 = 0$$

$$a_2 + 1 = 0$$

$$\boxed{a_2 = -1}$$

Step 4

$$\pi_2 = l^{-1} . v^{-1} . e^{-1} . \mu$$

$$\boxed{\pi_2 = \frac{\mu}{lve}}$$

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Now π_3,

$$\pi_3 = M^0 L^0 T^0 = L^{a_3} . (LT^{-1})^{b_3} . (ML^{-3})^{c_3} . K$$

$ML^{-1}T^{-2}$

K → change in pressure / volume strain $= \frac{MLT^{-2}/L^2}{L/L} = MLT^{-2}L^{-2} = ML^{-1}T^{-2}$

Power of $M = 0 = c_3 + 1 \rightarrow (1)$ $\therefore c_3 = -1$

" " $L = 0 = a_3 + b_3 - 3c_3 - 1 \rightarrow (2)$

$T = 0 = -b_3 - 2 \rightarrow (3)$ $\therefore b_3 = -2$

$c_3 = -1$, $b_3 = -2$ in (2)

$$a_3 - 2 - 3(-1) - 1 = 0$$

$$a_3 - 2 + 3 - 1 = 0$$

$$a_3 + 0 = 0$$

$$\boxed{a_3 = 0}$$

Step 4

$$\pi_3 = l^0 . v^{-2} . e^{-1} . K$$

$$\boxed{\pi_3 = \frac{K}{v^2 e} = \frac{K}{ev^2}}$$

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We know that,

$$f_1(\pi_1, \pi_2, \pi_3) = 0$$

$$f_1\left(\frac{R}{el^2v^2}, \frac{\mu}{lve}, \frac{K}{ev^2}\right) = 0$$

$$\frac{R}{el^2v^2} = \phi\left[\frac{\mu}{lve}, \frac{K}{ev^2}\right] \Rightarrow \boxed{R = el^2v^2\phi\left[\frac{\mu}{lve}, \frac{K}{ev^2}\right]}$$

Using Bukingham's π- theorem, show that the velocity through a circular orifice is given by $V = \sqrt{2gH}\,\phi\left[\frac{D}{H}, \frac{\mu}{\rho V H}\right]$, where H is the head causing flow, D is the diameter of the orifice, μ is co-efficient of viscosity, ρ is the mass density and g is the acceleration due to gravity.

Solution

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$$V = f(H, D, \mu, \rho, g)$$

$$f(V, H, D, \mu, \rho, g) = 0$$

Total no of variables 'n' = 6.

Total no of funda. dim 'm' = 3

$\therefore$ Total no. of dimensionless π-terms $= (n-m) = 6-3 = 3$.

Hence $f_1(\pi_1, \pi_2, \pi_3) = 0$

Selection of repeating variables:

H, g, ρ — repeating variables.

Equations:

$$\left.\begin{aligned} \pi_1 &= H^{a_1}.\,g^{b_1}.\,\rho^{c_1}.\,V \\ \pi_2 &= H^{a_2}.\,g^{b_2}.\,\rho^{c_2}.\,D \\ \pi_3 &= H^{a_3}.\,g^{b_3}\,\rho^{c_3}.\,\mu \end{aligned}\right\}$$

π-terms

$$\pi_1 = M^0L^0T^0 = L^{a_1}.(LT^{-2})^{b_1}.(ML^{-3})^{c_1}.\,LT^{-1}$$

$$(ML^{-3})^{c_1} \longrightarrow M^{c_1}L^{-3c_1}$$

Power of $M = 0 = c_1 \rightarrow (1)$

" " $L = 0 = a_1 + b_1 - 3c_1 + 1 \rightarrow (2)$

$T = 0 = -2b_1 - 1 \rightarrow (3)$ $\therefore b_1 = -\frac{1}{2}$

$c_1 = 0$ & $b_1 = -\frac{1}{2}$ in (2)

$2b_1 = -1$
$\Rightarrow b_1 = -\frac{1}{2}$

$a_1 - \frac{1}{2} + 1 = 0$

$$\boxed{a_1 = -\frac{1}{2}}$$

$$\pi_1 = H^{-\frac{1}{2}} . g^{-\frac{1}{2}} . \rho^0 . V^1$$

$$\boxed{\pi_1 = \frac{V}{\sqrt{gH}}}$$

π_2 term

$$\pi_2 = M^0 L^0 T^0 = L^{a_2} . (LT^{-2})^{b_2} . (ML^{-3})^{c} . L$$

Power of $M = 0 = c \rightarrow (1)$

" " $L = 0 = a_2 + b_2 - 3c + 1 \rightarrow (2)$

" " $T = 0 = -2b_2 . \rightarrow (3)$

$c = 0$ & $b = 0$ in (2)

$$a_2 + 1 = 0$$

$$\boxed{a_2 = -1}$$

$$\pi_2 = H^{-1} . g^0 . \rho^0 . D$$

$$\boxed{\pi_2 = \frac{D}{H}}$$

π_3 term,

$$\pi_3 = M^0 L^0 T^0 = L^{a_3} . (LT^{-2})^{b_3} . (ML^{-3})^{c} . ML^{-1}T^{-1}$$

Power of $M = 0 = +c + 1 \rightarrow (1)$ $\quad c = -1$

" " $L = 0 = a_3 + b_3 - 3c_3 - 1 \rightarrow (2)$

" " $T = 0 = -2b_3 - 1 \rightarrow (3)$; $b_3 = -\frac{1}{2}$

$c = 1$ & $b = -\frac{1}{2}$ in (2)

$$a_3 - \frac{1}{2} - 3 - 1 = 0$$

$$a_3 - \frac{1}{2} - 4 = 0$$

$$a_3 = 4 + \frac{1}{2}$$

$$a_3 = \frac{9}{2}$$

$$a_3 = -b_3 + 3c_3 + 1$$

$$= -\left(-\frac{1}{2}\right) - 3 + 1$$

$$= \frac{1}{2} - 2$$

$$= \frac{1-4}{2} = -\frac{3}{2}$$

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$$\Pi_3 = H^{-\frac{3}{2}} . g^{-\frac{1}{2}} . \rho^{-1} . \mu$$

$$\Pi_3 = H^{-1} . H^{-\frac{1}{2}} . g^{-\frac{1}{2}} . \rho^{-1} . \mu$$

$$\boxed{\Pi_3 = \frac{\mu}{H\rho\sqrt{Hg}}}$$

Multiplying by V
& Divide

$$\Pi_3 = \frac{\mu V}{H\rho V\sqrt{Hg}} \qquad \left[\because \Pi_1 = \frac{V}{\sqrt{Hg}}\right]$$

$$\boxed{\Pi_3 = \frac{\mu}{H\rho V} . \Pi_1}$$

Now,

$$f_1\left(\frac{V}{\sqrt{Hg}}, \frac{D}{H}, \frac{\mu}{H\rho V} . \Pi_1\right) = 0$$

$$\frac{V}{\sqrt{Hg}} = \phi\left[\frac{D}{H}, \Pi_1 . \frac{\mu}{H\rho V}\right]$$

$$\boxed{V = \sqrt{2gH}\ \phi\left[\frac{D}{H}, \frac{\mu}{\rho VH}\right]}$$

Using Bukingham's π - Theorem, show the discharge Q consumed by an oil ring is given by

$$Q = Nd^3\phi\left[\frac{\mu}{\rho Nd^2}, \frac{\sigma}{\rho N^2 d^3}, \frac{w}{\rho N^2 d}\right]$$

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where d is the internal diameter of the ring, N is rotational speed, ρ is density, σ is surface tension. and w is the specific ~~speed~~ weight of oil.

Solution:

$$Q = f(d, N, \rho, \sigma, \mu, w)$$

$$f(Q, d, N, \rho, \sigma, \mu, w) = 0$$

Total no. of variables $n = 7$

No. of fundamental dimensions $= m = 3$

No. of dimensionless π term $= n - m = 4$

$\therefore f_1 [\pi_1, \pi_2, \pi_3, \pi_4] = 0$

Selection of repeating variables:

d, N, ρ

$$\therefore \left.\begin{aligned} \pi_1 &= d^{a_1}, N^{b_1}, \rho^{c_1} . Q \\ \pi_2 &= d^{a_2}, N^{b_2}, \rho^{c_2} . \mu \\ \pi_3 &= d^{a_3}, N^{b_3}, \rho^{c_3} . \sigma \\ \pi_4 &= d^{a_4}, N^{b_4}, \rho^{c_4} . \omega \end{aligned}\right\} \text{equations}$$

π_1 term:

$$\pi_1 = M^0 L^0 T^0 = L^{a_1} . (T^{-1})^{b_1} . (ML^{-3})^{c_1} L^3 T^{-1}$$

Power of $M = 0 = c_1 \rightarrow (1) \quad \therefore c_1 = 0$

" " $L = 0 = a_1 - 3c_1 + 3 \rightarrow (2) \quad \therefore a_1 = -3$

" " $T = 0 = -b_1 - 1 \rightarrow (3) \quad \therefore b_1 = -1$

$$\pi_1 = d^{-3} . N^{-1} . \rho^0 . Q$$

$$\boxed{\pi_1 = \frac{Q}{d^3 N}}$$

π_2 term:

$$\pi_2 = M^0 L^0 T^0 = L^{a_2} . (T^{-1})^{b_2} . (ML^{-3})^{c_2} ML^{-1}T^{-1}$$

Power of $M = 0 = c_2 + 1 \rightarrow (1) \quad \therefore c_2 = -1$

" " $L = 0 = a_2 - 3c_2 - 1 \rightarrow (2) \quad \therefore a_2 = -2$

" " $T = 0 = -b_2 - 1 \rightarrow (3) \quad \therefore b_2 = -1$

$$\pi_2 = d^{-2} . N^{-1} . \rho^{-1} . \mu$$

$$\boxed{\pi_2 = \frac{\mu}{d^2 N . \rho}}$$

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π_3 term:

$$\pi_3 = M^0L^0T^0 = L^{a_3} \cdot (T^{-1})^{b_3} \cdot (ML^{-3})^{c_3} \cdot MT^{-2}$$

$MLT^{-2}L^{-1}$ $\frac{N}{m}$

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Power of $M = 0 = c_3 + 1 \rightarrow (1)$ $\therefore c_3 = -1$

" " $L = 0 = a_3 - 3c_3 \rightarrow (2)$ $a_3 = -3$

" " $T = 0 = -b_3 - 2 \rightarrow (3)$ $\therefore b_3 = -2$

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$$\pi_3 = d^{-3} \cdot N^{-2} \cdot \rho^{-1} \cdot \sigma$$

$$\boxed{\pi_3 = \frac{\sigma}{d^3 N^2 \rho}}$$

π_4 term:

$$\pi_4 = M^0L^0T^0 = L^{a_4} \cdot (T^{-1})^{b_4} (ML^{-3})^{c_4} \cdot ML^{-2}T^{-2}$$

Power of $M = 0 = c_4 + 1 \rightarrow (1)$ $\therefore c_4 = -1$

" " $L = 0 = a_4 - 3c_4 - 2 \rightarrow (2)$ $\therefore a_4 = -1$

" " $T = 0 = -b_4 - 2 \rightarrow (3)$ $\therefore b_4 = -2$

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$$\pi_4 = d^{-1} \cdot N^{-2} \cdot \rho^{-1} \cdot \omega$$

$$\boxed{\pi_4 = \frac{\omega}{d N^2 \rho}}$$

Substituting values of π's in Bukingham's π equation,

$$f_1(\pi_1, \pi_2, \pi_3, \pi_4) = f_1\left(\frac{Q}{d^3 N}, \frac{\mu}{d^2 N \rho}, \frac{\sigma}{d^3 N^2 \rho}, \frac{\omega}{d N^2 \rho}\right) = 0$$

$$\frac{Q}{Nd^3} = \phi_1 \left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{\rho N^2 d^3}, \frac{\omega}{\rho N^2 d}\right]$$

$$\boxed{Q = Nd^3 \cdot \phi \left[\frac{\mu}{\rho N d^2}, \frac{\sigma}{\rho N^2 d^3}, \frac{\omega}{\rho N^2 d}\right]}$$