



TOPIC : 2 - Tautology & Logical Equivalence

Tautology

A statement which is true always irrespective of the truth values of the individual variables is called a tautology.

Example $P \vee \neg P$ is a Tautology.

Contradiction

A statement which is always false is called a contradiction.

Example $P \wedge \neg P$ is a contradiction.

Contingency

A statement which is neither Tautology nor contradiction is called Contingency.

① Show that $Q \wedge \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

P	Q	$\neg P$	$\neg Q$	$P \wedge \neg Q$	$Q \vee (P \wedge \neg Q)$	$\neg P \wedge \neg Q$	S
T	T	F	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	T	T

\therefore Given statement is Tautology.



③ Show that $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is a tautology.

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$	$P \rightarrow R$	S
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence the given statement is a tautology.

④ Prove $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology.

P	Q	R	$\neg P$ (1)	$\neg Q$ (2)	$\neg R$ (3)	$P \vee Q$ (4)	$\neg Q \vee \neg R$ (5)	$\neg P \wedge \neg Q \vee \neg R$ (6)	$\neg(\neg P \wedge (\neg Q \vee \neg R))$ (7)	$(4) \wedge (7)$ (8)	$(1) \wedge (2) \vee (3)$ (9)	$(8) \vee (9)$ (10)
T	T	T	F	F	F	T	F	F	T	T	F	T
T	T	F	F	F	T	T	T	F	T	T	F	T
T	F	T	F	T	F	T	T	F	T	T	F	T
T	F	F	F	T	T	T	T	F	T	T	F	T
F	T	T	T	F	F	T	F	F	T	T	F	T
F	T	F	T	F	T	T	T	F	F	F	F	F
F	F	T	T	T	F	F	T	T	F	F	T	T
F	F	F	T	T	T	F	T	T	F	F	T	T



Equivalence

Two statements P and Q are equivalent iff $P \leftrightarrow Q$ or $P \rightleftharpoons Q$ is a tautology. It is denoted by the symbol $P \Leftrightarrow Q$ which is read as "P is equivalent to Q".

Logical Equivalences (or) Equivalence rules	
Idempotent Laws	$P \wedge P \Leftrightarrow P$ $P \vee P \Leftrightarrow P$
Associative Laws	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$
Commutative Laws	$P \wedge Q \Leftrightarrow Q \wedge P$ $P \vee Q \Leftrightarrow Q \vee P$
De Morgan's Laws	$\neg(P \wedge Q) \Leftrightarrow (\neg P \vee \neg Q)$ $\neg(P \vee Q) \Leftrightarrow (\neg P \wedge \neg Q)$
Distributive Laws	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
Complement Laws	$P \wedge \neg P \Leftrightarrow F$ $P \vee \neg P \Leftrightarrow T$
Absorption Laws	$P \vee (P \wedge Q) \Leftrightarrow P$ $P \wedge (P \vee Q) \Leftrightarrow P$
Contrapositive Law	$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
Conditional as disjunction	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$
Biconditional as conditional	$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

Ex ① show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$$\begin{aligned} & (\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \\ \Leftrightarrow & (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R) && (\because \text{Distributive law}) \\ \Leftrightarrow & ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R) && (\because \text{Associative law}) \\ \Leftrightarrow & [(\neg P \wedge \neg Q) \vee (Q \vee P)] \wedge R && (\because \text{Distributive law}) \\ \Leftrightarrow & [\neg(P \vee Q) \vee (P \vee Q)] \wedge R && (\because \text{De Morgan's law}) \\ \Leftrightarrow & T \wedge R && [\because P \vee \neg P \Leftrightarrow T] \\ \Leftrightarrow & R && [P \wedge T \Leftrightarrow P] \end{aligned}$$



⑧ Show that $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge (P \vee Q)) \wedge (\neg P \wedge (\neg P \wedge Q))$

$$\begin{aligned} &\Leftrightarrow (P \vee Q) \wedge ((\neg P \wedge \neg P) \wedge Q) \quad [\because \text{Associative Law}] \\ &\Leftrightarrow (P \vee Q) \wedge (\neg P) \wedge Q \quad [\because \text{Idempotent law}] \\ &\Leftrightarrow (P \wedge (\neg P \wedge Q)) \vee (Q \wedge (\neg P \wedge Q)) \quad [\because \text{Distributive law}] \\ &\Leftrightarrow ((P \wedge \neg P) \wedge Q) \vee (Q \wedge (\neg P \wedge Q)) \quad [\because \text{Associative law}] \\ &\Leftrightarrow (F \wedge Q) \vee (Q \wedge (\neg P \wedge Q)) \quad [\because \text{Commutative}] \\ &\Leftrightarrow F \vee (Q \wedge (\neg P \wedge Q)) \quad [\because \text{Associative}] \\ &\Leftrightarrow F \vee (Q \wedge \neg P) \quad [\because \text{Idempotent}] \\ &\Leftrightarrow Q \wedge \neg P \quad [\because \text{Associative}] \\ &\Leftrightarrow Q \wedge \neg P \quad [\because \text{Idempotent}] \end{aligned}$$

Tautological Implication

A statement P is said to be tautologically imply a statement Q iff $P \rightarrow Q$ is a tautology. we shall denote this idea by $P \Rightarrow Q$

⑨ Prove that $(P \rightarrow Q) \wedge (R \rightarrow Q) \Rightarrow (P \vee R) \rightarrow Q$

T.S.T $(P \rightarrow Q) \wedge (R \rightarrow Q) \rightarrow ((P \vee R) \rightarrow Q)$ is a tautology.

$$\begin{aligned} &(P \rightarrow Q) \wedge (R \rightarrow Q) \rightarrow ((P \vee R) \rightarrow Q) \\ &\Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q) \rightarrow (\neg(P \vee R) \vee Q) \quad [\because P \rightarrow Q \Leftrightarrow \neg P \vee Q] \\ &\Leftrightarrow ((\neg P \wedge \neg R) \vee Q) \rightarrow (\neg(P \vee R) \vee Q) \quad [\because \text{Distributive}] \\ &\Leftrightarrow (\neg(P \vee R) \vee Q) \rightarrow (\neg(P \vee R) \vee Q) \quad [\text{Demorgan's law}] \\ &\Leftrightarrow \neg(\neg(P \vee R) \vee Q) \vee (\neg(P \vee R) \vee Q) \quad [\because P \rightarrow Q \Leftrightarrow \neg P \vee Q] \\ &\Leftrightarrow T \quad [\neg P \vee P \Leftrightarrow T] \end{aligned}$$



③ Show that $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology.

Consider $(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))$

$$\Leftrightarrow (P \vee Q) \wedge \neg(\neg P \wedge \neg(Q \wedge R)) \quad [\because \text{Demorgan's Law}]$$

$$\Leftrightarrow (P \vee Q) \wedge \neg(\neg(P \vee (Q \wedge R))) \quad [\because \text{Demorgan's Law}]$$

$$\Leftrightarrow (P \vee Q) \wedge (P \vee (Q \wedge R)) \quad [\because \text{Double negation}]$$

$$\Leftrightarrow (P \vee Q) \wedge ((P \vee Q) \wedge (P \vee R)) \quad [\because \text{Distributive law}]$$

$$\Leftrightarrow P \vee (Q \wedge (Q \wedge R))$$

$$\Leftrightarrow P \vee ((Q \wedge Q) \wedge R) \quad [\because \text{Associative law}]$$

$$\Leftrightarrow P \vee (Q \wedge R) \quad [\because \text{Idempotent law}]$$

Now

$$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$$

$$\Leftrightarrow \neg(P \vee Q) \vee \neg(P \vee R) \quad [\text{Demorgan's law}]$$

$$\Leftrightarrow \neg((P \vee Q) \wedge (P \vee R)) \quad [\text{Demorgan's law}]$$

$$\Leftrightarrow \neg(P \vee (Q \wedge R)) \quad [\because \text{Distributive law}]$$

$$\text{Now } (P \vee (Q \wedge R)) \vee \neg(P \vee (Q \wedge R)) \Leftrightarrow T$$

Hence the gn. equation is a tautology.