



TOPIC : 10 – COMPLEX FORM

Formula: Complex form  $(-\pi, \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\text{where } c_n = \frac{1}{b-a} \int_a^b f(x) e^{-inx} dx$$

Formula: Complex form  $(-l, l)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i n \pi x}{l}}$$

$$\text{where } c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{i n \pi x}{l}} dx$$

① find the complex form  $f(x) = e^{ax}$ ,  $-\pi < x < \pi$

in the form  $e^{ax} = \frac{\sinh ax}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n a + in}{a^2 + n^2} e^{inx}$

& hence prove that  $\frac{\pi}{a \sinh a \pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2}$

sol:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{b-a} \int_a^b f(x) e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx$$



$$\begin{aligned}C_n &= \frac{1}{2\pi} \left[ \frac{e^{(a-in)\pi}}{a-in} - \frac{e^{(a-in)(-\pi)}}{a-in} \right] \\&= \frac{1}{2\pi(a-in)} \left[ e^{a\pi} \cdot e^{-in\pi} - e^{-a\pi} \cdot e^{in\pi} \right] \\&= \frac{1}{2\pi(a-in)} \left[ e^{a\pi} (\cos n\pi - i \sin n\pi) - e^{-a\pi} (\cos n\pi + i \sin n\pi) \right] \\&= \frac{1}{2\pi(a-in)} \left[ e^{a\pi} (-1)^n - e^{-a\pi} (-1)^n \right] \\&= \frac{(-1)^n}{2\pi(a-in)} (e^{a\pi} - e^{-a\pi}) \\&= \frac{(-1)^n}{\pi(a-in)} \frac{\sinh a\pi}{a+in} \\C_n &= \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)}\end{aligned}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)} e^{inx}$$

$$e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)} e^{inx}$$

Put  $x=0$ .

$$1 = \sum_{n=-\infty}^{\infty} \frac{(-1)^n (a+in) \sinh a\pi}{\pi(a^2+n^2)}$$



Equating the real parts, we get

$$\frac{\pi}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n a}{a^2 + n^2}$$

$$\therefore \frac{\pi}{a \sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2 + n^2}$$

2.  $f(x) = e^{ax}$  in  $(-l, l)$

Sol:  $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$

$$c_n = \frac{1}{2l} \int_{-l}^l f(x) e^{-\frac{inx\pi}{l}} dx$$

$$= \frac{1}{2l} \int_{-l}^l e^{ax} e^{-\frac{inx\pi}{l}} dx$$

$$= \frac{1}{2l} \int_{-l}^l e^{\left(a - \frac{in\pi}{l}\right)x} dx$$

$$= \frac{1}{2l} \left[ \frac{e^{\left(a - \frac{in\pi}{l}\right)x}}{a - \frac{in\pi}{l}} \right]_{-l}^l$$

$$= \frac{1}{2l} \left[ \frac{e^{\frac{al - in\pi}{l}} \cdot l}{a - \frac{in\pi}{l}} - \frac{e^{\frac{al - in\pi}{l} \cdot (-l)}}{a - \frac{in\pi}{l}} \right]$$

$$= \frac{1}{2} \left[ \frac{e^{al - in\pi} - e^{-al + in\pi}}{a - \frac{in\pi}{l}} \right]$$

$$C_n = \frac{1}{2l(al - in\pi)} \left[ e^{al} \cdot (-1)^n - e^{-al} \cdot (-1)^n \right]$$

$$= \frac{(-1)^n}{2l(al - in\pi)} \left[ e^{al} - e^{-al} \right]$$

$$C_n = \frac{(-1)^n \sinh al}{l(al - in\pi)}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh al}{l(al - in\pi)} e^{+ \frac{in\pi x}{l}}$$

③ Find the complex form of the fourier series of  $f(x) = e^{-x}$  in  $-1 < x < 1$ .

Sol:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$$

$$2l=2 \\ l=1$$

$$C_n = \frac{1}{b-a} \int_a^b f(x) e^{-\frac{in\pi x}{l}} dx$$

$$C_n = \frac{1}{b-a} \int_a^b f(x) e^{-\frac{in\pi x}{l}} dx$$

$$C_n = \frac{1}{2} \int_{-1}^1 e^{-x} \cdot e^{\frac{in\pi x}{1}} dx$$

$$= \frac{1}{2} \int_{-1}^1 e^{-(x + \frac{in\pi}{1})x} dx$$

$$= \frac{1}{2} \left[ \frac{e^{-(1+in\pi)x}}{-(1+in\pi)} \right]_{-1}^1$$





$$\begin{aligned}c_n &= \frac{1}{2(1+in\pi)} \left[ e^{+(1+in\pi)} - e^{-(1+in\pi)} \right] \\&= \frac{1}{2(1+in\pi)} \left[ e^1 (\cos n\pi + i \sin n\pi) - e^{-1} (\cos n\pi - i \sin n\pi) \right] \\&= \frac{1}{2(1+in\pi)} \left[ (-1)^n e - e^{-1} (-1)^n \right] \\&= \frac{(-1)^n}{2(1+in\pi)} \left[ e - e^{-1} \right] \\&= \frac{(-1)^n}{1+in\pi} \left( \frac{e - e^{-1}}{2} \right) \\c_n &= \frac{(-1)^n}{1+in\pi} \sinh 1 \\f(x) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+in\pi} \sinh 1 e^{in\pi x} \cdot \frac{1-in\pi}{1-in\pi} \\f(x) &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^2\pi^2} \sinh 1 e^{in\pi x} (1-in\pi)\end{aligned}$$