TOPIC : 7 - LAGRANGE'S EQUATION – METHOD OF GROUPING

(ii) Lagrange's Linear Equation:

Method of Grouping:

In the auxilary equation

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$ if the variables can be seperated in any pair of equations, then we get a solution of the form

$u(x,y) = a$ & $v(x,y) = b$.

(1) Solve: $px + qy = z$

Sol: Given $px + qy = z$

$P = x, \quad Q = y, \quad R = z$

The subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \log x = \log y + \log c_1$$

$$x = (y\, c_1)$$

$$\Rightarrow \frac{x}{y} = c_1 \quad \text{(ie) } u = \frac{x}{y}$$

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$$

$$y = z c_2$$

$$\Rightarrow \frac{y}{z} = c_2 \quad \text{(ie) } v = \frac{y}{z}$$

The solution of the given p.d.e is

$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0.$$

(2) Write the general integral of
$$pyz + qzx = xy.$$

Sol: The given eqn is of the form
$$Pp + Qq = R.$$
$$P = yz;\quad Q = zx,\quad R = xy$$
$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$
$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x\,dx = y\,dy$$
$$\Rightarrow \int x\,dx = \int y\,dy$$
$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + c_1$$
$$\Rightarrow c_1 = x^2 - y^2$$

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow y\,dy = z\,dz$$
$$\Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + c_2$$
$$\Rightarrow c_2 = y^2 - z^2$$
$$\therefore \phi(x^2 - y^2,\ y^2 - z^2) = 0.$$

(3) Solve: $x(y-z)p + y(z-x)q = z(x-y)$

Sol: The given eqn is of the form
$$Pp + Qq = R$$
$$P = x(y-z)\quad Q = y(z-x)\quad R = z(x-y)$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\frac{dx+dy+dz}{xy-xz+yz-xy+zx-zy} = \text{Each ratio}$$

$$dx+dy+dz=0.$$

Integrating, $x+y+z+c_1$

$$\text{Each ratio} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y-z+z-x+x-y}$$

$$\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

Integrating $\log x+\log y+\log z=\log c_2$

$$\log(xyz)=\log c_2$$

$$xyz=c_2$$

$$\therefore \phi(x+y+z,\ xyz)=0.$$

---

(4) $(x^2-yz)p+(y^2-zx)q=z^2-xy$.

Sol: This eqn is of the form $Pp+Qq=R$

$P=x^2-yz \quad Q=y^2-zx \quad R=z^2-xy$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx+dy+dz}{x^2-yz+y^2-zx+z^2-xy} = \frac{x\,dx+y\,dy+z\,dz}{x^3-xyz+y^3-xyz+z^3-xyz}$$

$$\frac{x\,dx + y\,dy + z\,dz}{x^3+y^3+z^3-3xyz} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$\frac{x\,dx + y\,dy + z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$x\,dx + y\,dy + z\,dz = (x+y+z)(dx+dy+dz)$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + c$$

$$x^2+y^2+z^2 = (x+y+z)^2 + c$$

$$x^2+y^2+z^2 = x^2+y^2+z^2+2xy+2yz+2zx+c$$

$$-2(xy+yz+zx) = c$$

$$xy+yz+zx = u \text{ (constant)}$$

$$\frac{dx-dy}{x^2-yz-y^2+zx} = \frac{dy-dz}{y^2-zx-z^2+xy}$$

$$\frac{d(x-y)}{(x^2-y^2)+z(x-y)} = \frac{d(y-z)}{y^2-z^2+x(y-z)}$$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(y+z+x)}$$

Integrating on both sides, we get

$$\log(x-y) = \log(y-z) + \log c$$

$$\log(x-y) - \log(y-z) = \log c$$

$$\log \frac{x-y}{y-z} = \log c$$

$$\frac{x-y}{y-z} = c.$$

The general solution is

$$\phi\left(xy+yz+zx, \frac{x-y}{y-z}\right) = 0.$$