



TOPIC : 6 – EQUATIONS REDUCIBLE TO STANDARD TYPES

Type - V

Form the equation of the type of

$$f(x^m, y^n, z) = 0$$

Case: (i) $m \neq 1, n \neq 1$ then $x = x^{1-m}, y = y^{1-n}$

Case: (ii) $m = n = 1$ then put $x = \log x$
 $y = \log y$

Next we follow type (3)

Type: b

Eqn of the type $f(z^m, z^n) = 0 \rightarrow (1)$

& $f_1(x, z^m) = f_2(y, z^n) \rightarrow (2)$

Case: (i) if $m \neq -1$ put $z = z^{m+1} \Rightarrow$ Type (1) \Rightarrow Type (4)

Case: (ii) if $m = -1$ then put $x = \log z$
 \rightarrow Type (1) \Rightarrow Type (4)



① Solve: $p^2 + x^2 y^2 q^2 = x^2 z^2$

Sol:

$$p^2 + x^2 y^2 q^2 = x^2 z^2$$

$$\div x^2, \quad \frac{p^2}{x^2} + y^2 q^2 = z^2$$

$$x^{-2} p^2 + y^2 q^2 = z^2$$

$$(x^{-1} p)^2 + (y q)^2 = z^2 \rightarrow \text{①}$$

This is of the form $f(x^m p, y^n q, z) = 0$

Here $m = -1, n = 1$

$$\text{put } X = x^{1-m} \quad Y = y^n \log y$$

$$X = x^{1+1} \quad Y = y^1 \log y$$

$$X = x^2 \quad Y = \log y$$

$$\frac{\partial X}{\partial x} = 2x \quad Q = \frac{\partial Z}{\partial Y}$$

$$P = \frac{\partial Z}{\partial X} \quad R = \frac{\partial Z}{\partial Y} = \frac{\partial Z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$$

$$\frac{\partial Z}{\partial x} = \frac{\partial Z}{\partial X} \cdot \frac{\partial X}{\partial x} \quad q = R \cdot \frac{1}{y}$$

$$p = 2x \cdot P \quad Yq = R$$

$$\frac{p}{2x} = P$$

$$x^{-1} p = 2P$$

$$\text{①} \Rightarrow (2P)^2 + R^2 = z^2 \rightarrow \text{②}$$

This is of the form $f(P, R, z) = 0$



We use Type (3)

$$\text{Let } u = x + ay$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$P = \frac{\partial z}{\partial x} = \frac{dz}{du} \frac{\partial u}{\partial x}$$

$$P = \frac{dz}{du}$$

$$Q = \frac{\partial z}{\partial y} = \frac{dz}{du} \frac{\partial u}{\partial y}$$

$$Q = a \frac{dz}{du}$$

$$\textcircled{2} \Rightarrow \left(2 \frac{dz}{du}\right)^2 + \left(a \frac{dz}{du}\right)^2 = z^2$$

$$(4 + a^2) \left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{4 + a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{4 + a^2}}$$

$$\frac{dz}{z} = \frac{1}{\sqrt{4 + a^2}} du$$

$$\int \frac{dz}{z} = \int \frac{1}{\sqrt{4 + a^2}} du$$

$$\log z = \frac{1}{\sqrt{4 + a^2}} u + b$$

$$\log z = \frac{1}{\sqrt{4 + a^2}} (x + ay) + b$$

$$\log z = \frac{1}{\sqrt{4 + a^2}} (x^2 + a \log y) + b$$



2. Solve: $x^2 p^2 + y^2 q^2 = z^2$

Sol:

$$x^2 p^2 + y^2 q^2 = z^2$$

$$(xp)^2 + (yq)^2 = z^2 \rightarrow \text{①}$$

This eqn is of the form $f(x^m, y^n, z) = 0$

Here $m=1, n=1$.

$$\text{Put } x = \log x \quad y = \log y$$

$$\frac{\partial x}{\partial x} = \frac{1}{x} \quad \frac{\partial y}{\partial y} = \frac{1}{y}$$

$$p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$p = p \cdot \frac{1}{x} \quad q = q \cdot \frac{1}{y}$$
$$xp = p \quad yq = q$$

Sub in eqn ①, we get

$$p^2 + q^2 = z^2 \rightarrow \text{②}$$

This eqn is of the form $f(p, q, z) = 0$

$$\text{Let } u = x + y$$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 1$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \quad q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}$$

$$p = \frac{dz}{du}$$

$$q = \frac{dz}{du}$$



Sub in (2) we get

$$\left(\frac{dy}{dx}\right)^2 + \left(a \frac{dz}{du}\right)^2 = z^2 \rightarrow (3)$$

$$\left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 = z^2$$

$$(1+a^2) \left(\frac{dz}{du}\right)^2 = z^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{z^2}{1+a^2}$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1+a^2}}$$

$$\frac{1}{z} dz = \frac{1}{\sqrt{1+a^2}} du$$

$$\int \frac{dz}{z} = \int \frac{1}{\sqrt{1+a^2}} du$$

$$\log z = \frac{1}{\sqrt{1+a^2}} u + c$$

$$\log z = \frac{1}{\sqrt{1+a^2}} (x+ay) + c$$

$$\log z = \frac{1}{\sqrt{1+a^2}} (\log x + a \log y) + c$$

which is the complete solution.



① Solve $z^2(p^2+q^2) = x^2+y^2$

Sol.

Given $z^2(p^2+q^2) = x^2+y^2$

$$(zp)^2 + (zq)^2 = x^2 + y^2 \rightarrow \text{①}$$

This eqn is of the form $f_1(x, z^m p) = f_2(y, z^m q)$

$$f_1(x, z^m p) = f_2(y, z^m q)$$

Here $m \neq -1$,

Put $z = z^{m+1}$

$$\Rightarrow z = z^{1+1} = z^2$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$p = 2zp$$

$$\frac{p}{2} = zp$$

Similarly, $\frac{q}{2} = zq$

Sub in eqn ①, we get

$$\left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 = x^2 + y^2$$

$$p^2 + q^2 = 4(x^2 + y^2)$$

$$p^2 - 4x^2 = -q^2 + 4y^2$$



This eqn is of the form $f_1(x, p) = f_2(y, Q)$

$$0 = (P \dots) \therefore p^2 - 4x^2 = 4y^2 - Q^2 = 4a^2$$
$$P^2 = 4a^2 + 4x^2 \quad Q^2 = -4a^2 + 4y^2$$
$$P = 2\sqrt{a^2 + x^2} \quad Q = 2\sqrt{y^2 - a^2}$$

$$dz = P dx + Q dy$$
$$dz = 2\sqrt{a^2 + x^2} dx + 2\sqrt{y^2 - a^2} dy$$
$$\int dz = 2 \int \sqrt{a^2 + x^2} dx + 2 \int \sqrt{y^2 - a^2} dy$$
$$z = 2 \left[\frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \left(\frac{x}{a} \right) + \frac{y}{2} \sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1} \left(\frac{y}{a} \right) \right] + b$$
$$z = x\sqrt{x^2 + a^2} + a^2 \sinh^{-1} \left(\frac{x}{a} \right) + y\sqrt{y^2 - a^2} - a^2 \cosh^{-1} \left(\frac{y}{a} \right) + b$$
$$= x\sqrt{x^2 + a^2} + y\sqrt{y^2 - a^2} + a^2 \left[\sinh^{-1} \left(\frac{x}{a} \right) - \cosh^{-1} \left(\frac{y}{a} \right) \right] + b$$



2. Solve : $p^2 + q^2 = z^2(x^2 + y^2)$

Sol: $p^2 + q^2 = z^2(x^2 + y^2) \rightarrow \text{---} \textcircled{1}$

$\left(\frac{p}{z}\right)^2 + \left(\frac{q}{z}\right)^2 = x^2 + y^2$

This eqn is of the form

$$f_1(x, z^m p) = f_2(y, z^n q)$$

Here $m = -1$.

put $z = \log z$.

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$P = \frac{1}{z} \cdot p$$

Similarly, $Q = \frac{1}{z} \cdot q$

Sub in eqn $\textcircled{1}$, we get

$$p^2 - x^2 = y^2 - a^2$$
 This eqn is of the form $f_1(x, p) = f_2(y, q)$ type (4)

$$p^2 - x^2 = y^2 - a^2 = a^2$$

$$p^2 - x^2 = a^2 \qquad y^2 - a^2 = a^2$$

$$p^2 = x^2 + a^2 \qquad a^2 = y^2 - a^2$$

$$p = \sqrt{x^2 + a^2} \qquad a = \sqrt{y^2 - a^2}$$

$$dz = p dx + Q dy$$

$$dz = \sqrt{x^2 + a^2} dx + \sqrt{y^2 - a^2} dy$$

$$\int dz = \int \sqrt{x^2 + a^2} dx + \int \sqrt{y^2 - a^2} dy$$

$$z = \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2} \sqrt{a^2 + x^2}$$

$$+ \frac{y}{2} \sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{y}{a}\right) + b.$$

($\because z = \log z$)

$$\log z = \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{x}{2} \sqrt{a^2 + x^2}$$

$$+ \frac{y}{2} \sqrt{y^2 - a^2} - \frac{a^2}{2} \cosh^{-1}\left(\frac{y}{a}\right) + b.$$