TOPIC : 6 – HALF RANGE SINE SERIES

Half range sine series in the interval $(0,\pi)$ $(0,l)$

Formula : $(0,\pi)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where } b_n = \frac{2}{b-a}\int_a^b f(x)\sin nx\,dx$$

Formula: $(0,l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\text{where } b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{n\pi x}{l}\,dx$$

Problems:

① $f(x) = x(\pi - x)$ in $0 < x < \pi$.

Sol:

$$f(x) = x(\pi - x)$$
$$= x\pi - x^2$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin nx\,dx$$

$u = x\pi - x^2$, $dv = \sin nx$, $v = \frac{-\cos nx}{n}$

$u_1 = \pi - 2x$, $v_1 = -\frac{\sin nx}{n^2}$

$u_2 = -2$, $v_2 = \frac{\cos nx}{n^3}$

$$= \frac{2}{\pi}\int_0^{\pi}(x\pi - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(x\pi - x^2)\frac{\cos nx}{n} + (\pi - 2x)\frac{\sin nx}{n^2} \mp 2\frac{\cos nx}{n^3}\right]_0^{\pi}$$

$$b_n = \cancel{8(-1)}\ \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

$$= \begin{cases} \frac{8}{\pi n^3} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=odd}^{\infty} \frac{8}{\pi n^3} \sin nx$$

$$f(x) = \frac{8}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n^3} \sin nx$$

Put $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=odd}^{\infty} \frac{\sin n\pi/2}{(2n-1)^3}$$

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{\pi}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{8} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

$$\frac{\pi^3}{32} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3}$$

(2) Obtain the sine series for the function

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{l}{2} \\ l-x, & \frac{l}{2} \le x \le l \end{cases}$$

Sol:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\ } \int^{b} f(x) \sin nx\, dx$$

$u = x \quad dv = \sin\frac{n\pi x}{l} \quad u = l - x$

$u_1 = 1 \quad v = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} \quad u_1 = -1$

$u_2 = 0 \quad v_1 = -\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} \quad u_2 = 0$

$$b_n = \frac{2}{l}\left[\left(-x\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)_0^{l/2} + \left(-(l-x)\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)_{l/2}^{l}\right]$$

$$b_n = \frac{2}{l}\left[-\frac{l}{2}\frac{1}{\frac{n\pi}{l}} + \left(\frac{l}{2}\right)\frac{1}{\frac{n\pi}{l}}\right]$$

$$b_n = \frac{2}{l}\left[\cancel{-\frac{l^2}{2}\frac{\cos\frac{n\pi}{2}}{n\pi}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2\sin n\pi}{n^2\pi^2} \cancel{+ \frac{l^2}{2n\pi}\cos\frac{n\pi}{2}} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{l}\left(\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$b_n = \frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=1}^{\infty}\frac{4l}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$$

(3) $f(x) = (2-x)$ in $(0, 2)$.

Sol:

$$b_n = \frac{2}{2}\int_0^2 (2-x)\sin\frac{n\pi x}{2}\,dx$$

$u = 2-x \qquad dv = \sin\frac{n\pi x}{2}$

$$v = -\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}$$

$u_1 = -1 \qquad v_1 = -\frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}$

$$b_n = \left[-(2-x)\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}} - \frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= \frac{2 \cdot 2}{n\pi}$$

$$= \frac{4}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\frac{n x\pi}{2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{n^2\pi}\left[-1+(-1)^n\right]$$

$$a_n = \begin{cases} \frac{-4}{n^2\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[1-(-1)^n\right]\cos nx$$

$$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} \frac{-4}{n^2\pi}\cos nx$$

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2. Find the half range cosine series of the function $f(x) = x(\pi - x)$ in $(0,\pi)$.

Sol: $f(x) = x\pi - x^2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(x\pi - x^2)\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\pi - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$