



Method of multipliers:

choose any three multipliers l, m, n which may be constants or functions of x, y, z , we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

It is possible to choose l, m, n such that

$$lP + mQ + nR = 0 \text{ then } l dx + m dy + n dz = 0$$

If $l dx + m dy + n dz$ is an exact differential then on integration we get a solution

$$u = a.$$

The multipliers l, m, n are called Lagrangian multipliers.



① solve: $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)r$

Sol: Given $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)r$

This is of the form $Pp + Qq = R$.

where $P = x(y^2 - z^2)$ $Q = y(z^2 - x^2)$ $R = z(x^2 - y^2)$

The subsidiary eqns are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

Each ratio = $\frac{x dx + y dy + z dz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)}$

$$= \frac{x dx + y dy + z dz}{x^2 y^2 - x^2 z^2 + y^2 z^2 - y^2 x^2 + z^2 x^2 - z^2 y^2}$$

$$\Rightarrow x dx + y dy + z dz = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

Each ratio = $\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2}$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c$$



2. Solve: $(mz - ny)p + (nx - lz)q = ly - mx$

Sol:

Given $(mz - ny)p + (nx - lz)q = ly - mx$

This eqn is of the form $Pp + Qq = R$

where $P = mz - ny$, $Q = nx - lz$, $R = ly - mx$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}$$

$$\text{Each ratio} = \frac{dx + dy + dz}{mz - ny + nx - lz + ly - mx}$$

$$= \frac{x dx + y dy + z dz}{x mz - x ny + ny x - y lz + ly z - m x z}$$

$$\Rightarrow x dx + y dy + z dz = 0$$

Integrating.

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$\text{Each ratio} = \frac{ldx + mdy + ndz}{lmz - lny + mnx - mlz + lny - mnx}$$



$\Rightarrow ldx + mdy + ndz = 0$

Integrating,

$$lx + my + nz = C$$

The general solution is

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, lx + my + nz\right) = 0.$$

③ Solve: $(3z - 4y)p + (4x - 2z)q = 2y - 3x$

Sol:

This eqn is of the form $Pp + Qq = R$

$$P = 3z - 4y, \quad Q = 4x - 2z, \quad R = 2y - 3x$$
$$\frac{dx}{3z - 4y} = \frac{dy}{4x - 2z} = \frac{dz}{2y - 3x}$$

Each ratio = $\frac{x dx + y dy + z dz}{3xz - 4yx + 4xy - 2yz + 2yz - 3xz}$

$$\Rightarrow x dx + y dy + z dz = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C$$

Each ratio = $\frac{2dx + 3dy + 4dz}{6z - 8y + 12x - 6z + 8y - 12x}$

$$\Rightarrow 2dx + 3dy + 4dz = 0.$$



$$2x + 3y + 4z = c_2$$

The general solution is

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, 2x + 3y + 4z\right) = 0$$

④ Solve: $(y - xz)p + (yz - x)q = (x + y)(x - y)$

Sol:

This eqn is of the form $Pp + Qq = R$

$$P = y - xz, \quad Q = yz - x, \quad R = (x + y)(x - y)$$

$$\frac{dx}{y - xz} = \frac{dy}{yz - x} = \frac{dz}{(x + y)(x - y)} = \frac{dz}{x^2 - y^2}$$

$$\text{Each ratio} = \frac{x dx + y dy + z dz}{xy - x^2z + y^2z - xy + x^2z - zy^2}$$

$$\Rightarrow x dx + y dy + z dz = 0$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$\Rightarrow u = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}$$

$$\text{Each ratio} = \frac{y dx + x dy + dz}{y^2 - xyz + xyz - x^2 + x^2 - yz}$$

$$y dx + x dy + dz = 0$$

Integrating,

$$\frac{y^2}{2} + \frac{x^2}{2} + z = -$$



$d(xy) + dz = 0$

on integration, we get

$$xy + z = c,$$
$$V = xy + z.$$

The general solution is

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, xy + z\right) = 0.$$