TOPIC: 5 - SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Type : 3 $F(z, p, q) = 0$.

This eqn is of the form $f(z, p, q) = 0$ — (1)

Let $z = f(x+ay)$ be the solution of (1)

put $x + ay = u$ — (2)

Then $z = f(u)$ — (3)

Substitute $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$ Then

Integrating we get the solution.

1. Solve: $p(1+q) = qz$.

Sol:

Given $p(1+q) = qz$ — (1)

This eqn is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

$$(1) \Rightarrow \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du}$$

$$\therefore 1 + a\frac{dz}{du} = az$$

$$a\frac{dz}{du} = az - 1$$

$$\frac{dz}{du} = \frac{az-1}{a}$$

$$\frac{du}{dz} = \frac{a}{az-1}$$

$$du = \frac{a}{az-1}dz$$

Integrating on both sides, we get

$$u = \log(az-1) + b.$$

$$x + ay = \log(az-1).$$

(2) $z^2 = 1 + p^2 + q^2$.

Sol:

Given $z^2 = 1 + p^2 + q^2$ — (1)

This eqn is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du} \quad q = a\frac{dz}{du}$$

$$(1) \Rightarrow z^2 = 1 + \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 (1 + a^2) = z^2 - 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2 - 1}{a^2 + 1}$$

$$\Rightarrow \frac{dz}{du} = \sqrt{\frac{z^2 - 1}{a^2 + 1}}$$

$$\Rightarrow \sqrt{a^2 + 1}\, dz = \sqrt{z^2 - 1}\, du$$

$$\Rightarrow \sqrt{a^2 + 1}\, \frac{dz}{\sqrt{z^2 - 1}} = du$$

Integrating,

$$\sqrt{a^2 + 1} \int \frac{dz}{\sqrt{z^2 - 1}} = \int du + b$$

$$\sqrt{a^2 + 1}\, \cosh^{-1}(z) = u + b$$

$$\sqrt{a^2 + 1}\, \cosh^{-1}(z) = x + ay + b$$

is the complete solution.

(3) $p+q=z$.

Sol:

Given $p+q=z \rightarrow (1)$

Let $u=x+ay$

$$p=\frac{dz}{du}, \quad q=a\frac{dz}{du}$$

$$\frac{dz}{du}+a\frac{dz}{du}=z$$

$$\frac{dz}{du}(1+a)=z$$

$$\frac{dz}{du}=\frac{z}{1+a}$$

$$(1+a)\frac{dz}{z}=du$$

Integrating,

$$(1+a)\int\frac{dz}{z}=\int du$$

$$(1+a)\log z=u+b$$

$(1+a)\log z=x+ay+b$. is the complete solution.

(4) Solve: $p(1-q^2) = q(1-z)$

Sol:

$$\text{Gn} \quad p(1-q^2) = q(1-z) \rightarrow (1)$$

Let $u = x + ay$

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

$$\frac{dz}{du}\left(1 - a^2\left(\frac{dz}{du}\right)^2\right) = a\frac{dz}{du}(1-z)$$

$$1 - a^2\left(\frac{dz}{du}\right)^2 = a \cancel{\frac{dz}{du}} (1-z).$$

$$= a - az$$

$$a^2\left(\frac{dz}{du}\right)^2 = -a + az + 1$$

$$a\frac{dz}{du} = \sqrt{1-a+az}$$

$$\frac{a}{\sqrt{1-a+az}}\,dz = du$$

$$a(1-a+az)^{-\frac{1}{2}}\,dz = du$$

integrating we get

$$\frac{a(1-a+az)^{\frac{1}{2}}}{\frac{1}{2}a} = u + b$$

$$2(1-a+az)^{\frac{1}{2}} = x + ay + b$$ is the complete solution.

Type:(iv)

Equation containing x, y, p, q.

i) Attach x & p in one side

ii) Attach y & q in other side

iii) Let it be equal to k

iv) Find p & q

v) $dz = p\,dx + q\,dy$

vi) Integrate we get the complete solution.

① Solve: $p+q = x+y$

Sol: Gn

$$p - x = y - q = k$$

$$p - x = k, \quad y - q = k$$

$$p = x + k \qquad q = y - k$$

$$dz = p\,dx + q\,dy$$

$$dz = (x+k)\,dx + (y-k)\,dy$$

Integrating,

$z = \frac{x^2}{2} + kx + \frac{y^2}{2} - ky + b$ is the complete solution

Diff p.w.r.to b, $0 = 1$ is absurd

There is no. singular solution

② Solve: $pq = xy$

Sol:

$$pq = xy$$

$$\frac{p}{x} = \frac{q}{q} = k \text{ (say)}$$

$$\frac{p}{x} = k, \quad \frac{y}{q} = k.$$

$$p = xk, \quad q = \frac{y}{k}$$

$$dz = p\,dx + q\,dy$$

$$= (xk)dx + \left(\frac{y}{k}\right)dy$$

Integrating we get

$$z = \frac{x^2}{2}k + \frac{y^2}{2k} + b \text{ is the}$$

complete solution.

Diff p.w.r.to b, $0 = 1$ is absurd

There is no singular integral.

③ Solve: $p^2 y(1+x^2) = qx^2$

Sol: Gn $p^2 y(1+x^2) = qx^2$

$$\Rightarrow \frac{p^2(1+x^2)}{x^2} = \frac{q}{y} = k$$

$$\frac{p^2(1+x^2)}{x^2} = k \qquad \frac{q}{y} = k$$

$$p^2 = \frac{kx^2}{1+x^2} \qquad q = yk.$$

$$p = \frac{\sqrt{k}\,x}{\sqrt{1+x^2}}$$

$$dz = p\,dx + q\,dy$$

$$dz = \frac{\sqrt{k}\,x}{\sqrt{1+x^2}}\,dx + yk\,dy$$

Integrate.

$$z = \sqrt{k}\int \frac{x}{\sqrt{1+x^2}}\,dx + k\int y\,dy + b$$

$z = \sqrt{k}\,\sqrt{1+x^2} + k\frac{y^2}{2} + b$ is the complete solution.

(4) $\sqrt{p} + \sqrt{q} = x + y$

Sol:

$$\sqrt{p} + \sqrt{q} = x + y$$

$$\sqrt{p} - x = y - \sqrt{q} = k$$

$$\sqrt{p} - x = k \qquad y - \sqrt{q} = k$$

$$\sqrt{p} = x + k \qquad \sqrt{q} = y - k$$

$$p = (x+k)^2 \qquad q = (y-k)^2$$

$$dz = p\,dx + q\,dy$$

$$dz = (x+k)^2\,dx + (y-k)^2\,dy$$

Integrate, $z = \frac{(x+k)^3}{3} + \frac{(y-k)^3}{3} + b$ is the complete solution.