TOPIC: 3 - SOLUTION OF STANDARD TYPES OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Define : Singular Integral

Let $f(x, y, z, p, q) = 0. \rightarrow (1)$

Let the complete integral be

$$\phi(x, y, z, a, b) = 0 \rightarrow (2)$$

Diff (2) p.w.r.to a & b in turn we get

$$\frac{\partial \phi}{\partial a} = 0 \rightarrow (3) \text{ and}$$

$$\frac{\partial \phi}{\partial b} = 0 \rightarrow (4)$$

The elimination of a & b from the three equations (2), (3) & (4) if it exists, is called the singular Integral.

Type: 1 $f(p, q) = 0$.

[The equations contain p and q only]

Suppose that $z = ax + by + c$ is a trial solution of $f(p, q) = 0$.

where $p = a$, $q = b$ we get $f(a, b) = 0$

Here a & b are the constant.

Eliminate any one constant we get the complete solution.

1. Find the complete solution of $\sqrt{p}+\sqrt{q}=1$

Sol: Given $\sqrt{p}+\sqrt{q}=1. \rightarrow (1)$

This equation of the form $f(p,q)=0$.

Hence the trial solution is $z=ax+by+c \rightarrow (2)$

where $p=a$ & $q=b$.

Substitute in eqn (1) we get

$$\sqrt{a}+\sqrt{b}=1$$

$$\Rightarrow \sqrt{b}=1-\sqrt{a} \Rightarrow b=(1-\sqrt{a})^2$$

$$\therefore z=ax+(1-\sqrt{a})^2 y+c.$$

2. $p+q=pq$.

Sol: Given $p+q=pq \rightarrow (1)$

This equation of the form $f(p,q)=0$

Hence the trial solution is $z=ax+by+c \rightarrow (2)$

where $p=a$ & $q=b$

Substitute in eqn (1), we get

$$a+b=ab$$

$$b-ab=a$$

$$\Rightarrow b(1-a)=a$$

$$b=\frac{a}{1-a}$$

The complete solution is $z=ax+\left(\frac{a}{1-a}\right)y+c$.

(3) $p^2+q^2=npq$.

Sol: Given $p^2+q^2=npq$.

This eqn is of the form $z=ax+f(p,q)=0$

Hence the trial solution is $z=ax+by+c$

where $p=a$ & $q=b$

$$a^2+b^2=nab$$

$$b^2-nab+a^2=0$$

$$b=\frac{na\pm\sqrt{a^2n^2-4a^2}}{2}$$

$$=\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]$$

The complete solution is

$$z=ax+\frac{a}{2}\left[n\pm\sqrt{n^2-4}\right]y+c$$

(6)

(4) $p - 3q = 6$

Sol: Given $p - 3q = 6$

This eqn of the form $f(p,q) = 0$

Hence the trial solution is $z = ax + by + c$

where $p = a$ & $q = b$

$$a - 3b = 6.$$

$$\Rightarrow 3b = 6 - a$$

$$\Rightarrow b = \frac{6-a}{-3} = -2 + \frac{a}{3}.$$

The complete solution is

$$z = ax + \left(-2 + \frac{a}{3}\right)y + c.$$

(5) $p - q = 0.$

Sol: Given $p - q = 0.$

This eqn of the form $f(p,q) = 0$

Hence the trial solution is $z = ax + by + c \rightarrow (2)$

Sub. (2) in (1). Here $p = a$ & $q = b$

$$a - b = 0.$$

$$b = a.$$

The complete solution is

$$z = ax + ay + c = a(x+y) + c$$

Type : 2 Clairaut's form

$$z = px + qy + f(p,q).$$

This eqn of the form $z = px + qy + f(p,q)$.

$\therefore$ The complete integral is

$$z = ax + by + f(a,b)$$

To find the singular integral

Diff p.w.r. to a & b.

we get the solution in terms of x, y, z.

To find the general solution

put $b = f(a)$

Eliminate 'a' we get the general solution.

1. Solve : $z = px + qy + pq$.

Sol: Given $z = px + qy + pq \rightarrow (1)$

This eqn is of the form. $z = px + qy + f(p,q) \rightarrow (2)$

$\therefore$ The complete integral is

$$z = ax + by + f(a,b)$$

$$z = ax + by + ab.$$

To find singular integral

Diff p.w.r.to a & b.

$$\frac{\partial z}{\partial a} = 0 \Rightarrow x + b = 0$$

$$\Rightarrow b = -x$$

$\frac{\partial z}{\partial b} = 0 \Rightarrow y + a = 0$

$\Rightarrow a = -y.$

$\therefore z = (-y)x + (-x)y + (-y)(-x)$

$= -xy - xy + xy$

$z = -xy$

$z + xy = 0.$

which is a singular solution.

To get the general integral

put $b = f(a)$ in eqn ①.

$z = ax + f(a)y + af(a) \rightarrow ③$

Diff p.w.r.to a, $\frac{\partial z}{\partial a} = 0.$

$\Rightarrow x + f'(a)y + a f'(a) + f(a) = 0 \rightarrow ⑤$

Eliminate 'a' between ④ & ⑤ we get the general solution.

② $z = px + qy + p^2 - q^2$

Sol:

Given $z = px + qy + p^2 - q^2$ — ①

This eqn of the form $z = px + qy + f(p,q)$ — ②

The complete integral is

$z = ax + by + f(a,b)$

(i) $z = ax + by + a^2 - b^2$ — ③

To find singular integral

Diff p.w.r.to a & b.

$\frac{\partial z}{\partial a} = 0 \Rightarrow x + 2a = 0$

$2a = -x$

$a = \frac{-x}{2}$.

$\frac{\partial z}{\partial b} = 0 \Rightarrow y - 2b = 0$.

$\Rightarrow y = 2b$

$\Rightarrow b = \frac{y}{2}$.

Sub a, b in ②.

$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$.

$= \frac{-2x^2 + 2y^2 + x^2 - y^2}{4}$

$z = \frac{-x^2 + y^2}{4}$

$4z = y^2 - x^2$ is the singular integral

To find the general integral

Put $b = f(a)$ in ②

$z = ax + f(a)y + a^2 - (f(a))^2$ → ④

$\frac{\partial z}{\partial a} = 0$

$\Rightarrow x + f'(a)y + 2a - 2f(a).f'(a) = 0$ → ⑤

Eliminate 'a' between ④ & ⑤ we get

the general solution.

(5) Solve: $z = px + qy + \sqrt{p^2+q^2+1}$ (b)

Sol:

Given $z = px + qy + \sqrt{p^2+q^2+1}$

This eqn is of the form $z = px + qy + f(p,q)$

∴ The complete integral is

$z = ax + by + f(a,b)$

(ie) $z = ax + by + \sqrt{a^2+b^2+1} \longrightarrow (1)$

To find : singular integral

Diff p.w.r.to a & b,

$\frac{\partial z}{\partial a} = 0 \Rightarrow x + \frac{1}{2}(a^2+b^2+1)^{-\frac{1}{2}} \cdot 2a = 0$

$\Rightarrow x + \frac{a}{\sqrt{a^2+b^2+1}} = 0$

ie. $x = \frac{-a}{\sqrt{a^2+b^2+1}}$ —(2)

$\frac{\partial z}{\partial b} = 0 \Rightarrow y + \frac{1}{2}(a^2+b^2+1)^{-\frac{1}{2}} \cdot 2b = 0$

$\Rightarrow y + \frac{b}{\sqrt{a^2+b^2+1}} = 0$

$\Rightarrow y = \frac{-b}{\sqrt{a^2+b^2+1}}$ —(3)

$x^2+y^2 = \frac{a^2}{a^2+b^2+1} + \frac{b^2}{a^2+b^2+1}$

$= \frac{a^2+b^2}{a^2+b^2+1}$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{a^2+b^2+1}$$

$$1-x^2-y^2 = \frac{a^2+b^2+1-a^2-b^2}{a^2+b^2+1}$$

$$1-x^2-y^2 = \frac{+1}{a^2+b^2+1}$$

$$\sqrt{1-x^2-y^2} = \frac{1}{\sqrt{a^2+b^2+1}}$$

$$\sqrt{a^2+b^2+1} = \frac{1}{\sqrt{1-x^2-y^2}}$$

(2) $\Rightarrow$ $x = -a\sqrt{1-x^2-y^2} \Rightarrow a = \dfrac{-x}{\sqrt{1-x^2-y^2}}$

(3) $\Rightarrow$ $y = -b\sqrt{1-x^2-y^2} \Rightarrow b = \dfrac{-y}{\sqrt{1-x^2-y^2}}$

Sub in (1)

$$z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$x^2+y^2+z^2 = 1$ is the singular solution.

To get the general integral

put $b = f(a)$ in ①.

$$z = ax + f(a)y + \sqrt{1+a^2+(f(a))^2} \rightarrow ④$$

Diff ④ p.w.r.to 'a',

$$0 = x + f'(a)y + \frac{1}{2}\left(1+a^2+(f(a))^2\right)^{-\frac{1}{2}}(2a + 2f(a).f'(a))$$

$$0 = x + f'(a)y + \frac{a + f(a)f'(a)}{\sqrt{1+a^2+(f(a))^2}} \rightarrow ⑤$$

Eliminate 'a' between ④ & ⑤ we get the general solution.

④ $z = px + qy - 2\sqrt{pq}$

Sol:

This eqn is of the form

$$z = px + qy + f(p, q)$$

The complete integral is

$$z = ax + by + f(a,b)$$

$$z = ax + by - 2\sqrt{ab} \quad — ①$$

To find singular integral.

Diff p.w.r.to a & b in ①

$$\frac{\partial z}{\partial a} = 0$$

$$\Rightarrow x + 0 - 2\tfrac{1}{2}(ab)^{-\frac{1}{2}}b = 0$$

$$\Rightarrow x = (ab)^{-\frac{1}{2}}.b$$

$$\frac{\partial z}{\partial b} = 0$$

$$\Rightarrow y - 2\cdot\frac{1}{2}(ab)^{-\frac{1}{2}}\cdot a = 0$$

$$\Rightarrow y = (ab)^{-\frac{1}{2}}\cdot a$$

$$\Rightarrow xy = (ab)^{-\frac{1}{2}}\cdot(ab)^{-\frac{1}{2}}\cdot ab$$

$$= a^{-\frac{1}{2}}\, b^{-\frac{1}{2}}\, a^{-\frac{1}{2}}\cdot b^{-\frac{1}{2}}\cdot a\cdot b$$

$$= a^{-1}\cdot b^{-1}\cdot a\cdot b$$

$$xy = 1$$