3). Maximum principal strain theory [saint vanant's theory]. 41 (21)

Acc. to this theory, the failure (or) yielding occurs at a point in a member when the maximum principal (or) (normal) strain in a bi-axial system reaches the limiting value of strain (i.e, strain at yield point) as determined from a simple tensile test.

$$\left.\begin{array}{l}\sigma_1 - \gamma(\sigma_2+\sigma_3) \text{ (or) } \sigma_2 - \gamma(\sigma_3+\sigma_1) \text{ (or)} \\ \sigma_3 - \gamma(\sigma_1+\sigma_2)\end{array}\right\} = \sigma_y$$

whichever is max

γ = Poission's ratio

$$\boxed{e = \frac{\sigma_{t1}}{E} - \frac{\sigma_{t2}}{mE} = \frac{\sigma_y}{E\,FOS}}$$

(or) if E is not given.

$$\boxed{\sigma_1 - \gamma[\sigma_2+\sigma_3] = \frac{\sigma_y}{FOS}}$$

4). Maximum strain energy theory (Haigh's theory).

Acc. to this theory, failure occurs when strain energy stored per unit volume of the stressed element becomes equal to the strain energy stored per unit volume in the tension specimen at the yield point.

$$\boxed{\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\gamma(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right] = \left[\frac{\sigma_y}{FOS}\right]^2 \frac{1}{2E}}$$

5). Distortion energy theory (Octahedral theory).

Acc. to this theory, failure occurs when the strain energy of distortion (deformation) per unit volume of the component becomes equal to the strain energy of distortion per unit volume of the tension test specimen.

$$\sigma_1^2+\sigma_2^2+\sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 = \sigma_y^2.$$

2). A cylindrical shaft made of steel of yield strength 700 Mpa is subjected to static loads consisting of bending moment 10 kN-m and a torsional moment 30 kN-m. Determine the diameter of the shaft using two different theories of failure, and assuming a factor of safety of 2. Take E = 210 Gpa and Poisson's ratio = 0.25.

Solution:-

σ_y = yield strength = 700 Mpa = 700 N/mm^2.

B.M = M = 10 kN-m = 10×10^6 N-mm.

T = 30 kN-m = 30×10^6 N-mm.

F.S = 2.

$E = 210\,GPa \Rightarrow 2\times10^5\,N/mm^2 = 210\times10^3\,N/mm^2$

$1/m = \gamma = 0.25$.

Bending (tensile) stress due to the bending moment,

$$\sigma_x = \frac{M}{Z} \qquad Z = \frac{I}{y} = \frac{\frac{\pi}{64}d^4}{d/2} = \frac{\pi}{32}d^3.$$

$$\sigma_x = \frac{M}{Z} = \frac{10\times10^6}{0.0982 d^3}\ N/mm^2. \qquad Z = \pi/32 \times d^3 = 0.0982 d^3\ mm^3$$

$$\boxed{\sigma_x = \frac{101.8\times10^6}{d^3}\ N/mm^2}$$

Shear stress due to torsional moment

$$\frac{T}{J} = \frac{\tau}{r}$$

$$\frac{T}{J} = \frac{\tau}{r}.$$

$$\tau = \frac{T \cdot r}{J} \Rightarrow \frac{T \times \cancel{d}/\cancel{2}}{\pi/\cancel{32}_{16} \times d^{\cancel{4}\,3}}$$

$$\tau_{xy} = \tau_x = \frac{T \times 16}{\pi \times d^3} \Rightarrow \frac{30 \times 10^6 \times 16}{\pi d^3} = \frac{152.8 \times 10^6}{d^3} \; N/mm^2$$

Max. principle stress.

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2}.$$

$$\sigma_1 = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{(\sigma_x)^2 + 4\tau_{xy}^2} \qquad [\sigma_y = 0].$$

$$\sigma_1 = \frac{101.8 \times 10^6}{2d^3} + \frac{1}{2}\sqrt{\left[\frac{101.8 \times 10^6}{d^3}\right] + 4\left[\frac{152.8 \times 10^6}{d^3}\right]^2}$$

$$= \frac{50.9 \times 10^6}{d^3} + \frac{1}{2} + \frac{10^6}{d^3}\left[\sqrt{(101.8)^2 + 4[152.8)^2}\right].$$

$$= \frac{50.9 \times 10^6}{d^3} + \frac{161 \times 10^3}{d^3} = \frac{211.9 \times 10^6}{d^3} \; N/mm^2$$

Minimum principle stress

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{(\sigma_1 - \sigma_2)^2 + 4\tau^2}.$$

$$\sigma_2 = \frac{50.9 \times 10^6}{d^3} - \frac{161 \times 10^6}{d^3} = -\frac{110.1 \times 10^6}{d^3} \; N/mm^2$$

Let us now find the diameter of shaft (d) by considering the max. shear stress theory and max. strain energy theory.

1) According to max. shear stress theory

Max. Shear stress theory,

$$\tau_{max} = \frac{\tau_{yt}}{F_{os}}$$ ← shear for yield point

$$\cancel{\tau_{max} = \frac{\sigma_1 - \sigma_2}{2}}$$

$$(\sigma_1 - \sigma_2) \text{ (or) } (\sigma_2 - \sigma_3) \text{ (or) } (\sigma_3 - \sigma_1) = \frac{\sigma_y}{n \text{ (or) } F_{os}}$$

$$\boxed{\sigma_1 - \sigma_2 = \frac{\sigma_y}{F_{os}}}$$

$$\frac{211.9\times10^6}{d^3} - \left[-\frac{110.1\times10^6}{d^3}\right] = \frac{700}{2}$$

$$\frac{211.9\times10^6 + 110.1\times10^6}{d^3} = 350$$

$$\frac{322\times10^6}{d^3} = 350$$

$$d^3 = 920\times10^3$$

$$\boxed{d = 97.25\,mm}$$

2) According to max. strain energy theory.

$$\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\gamma\left[\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right]\right] = \left[\frac{\sigma_y}{F_{os}}\right]^2\frac{1}{2E}$$

(or)

$$\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\gamma\left[\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\right] = \frac{\sigma_y}{F_{os}}$$

$$\left[\frac{211.9\times10^6}{d^3}\right]^2 + \left[\frac{-110.1\times10^6}{d^3}\right]^2 - 2\times0.25\left[\frac{211.9\times10^6}{d^3}\times\frac{-110.1\times10^6}{d^3}\right] = \left[\frac{700}{2}\right]^2$$

$$(\text{or}) = \frac{44902\times10^{12}}{d^6} + \frac{12122\times10^{12}}{d^6} + \frac{11665\times10^{12}}{d^6} = 122500$$

$$\frac{68689\times10^{12}}{d^6} = 122500$$

$$d^6 = 68689\times10^{12}/122500$$

$$\boxed{d = 90.8\,mm}$$

3). A mild steel shaft of 50mm diameter is subjected to a bending moment of 2000N-m and a torque T. If the yield point of the steel in tension is 200 MPa. Find the max. value of this torque without causing yielding of the shaft according to

i). The maximum principle stress.

ii) The maximum shear stress

iii). the maximum distortion strain energy energy theory of yielding

Given:-

$d = 50mm$

$M = 2000N\text{-}m = 2\times10^6 N\text{-}mm$.

σ_y (or) yield Stress $= 200\,MPa = 200\,N/mm^2$.

Let T = Max. torque without causing yielding of the shaft, in N-mm.

Solution:-

1) According to maximum principle stress theory.

Bending stress due to bending moment.

$$\sigma_x = \frac{M}{Z}$$

$$\sigma_x = \frac{M}{Z} = \frac{2\times10^6}{12273}$$

$$\boxed{\sigma_x = 163\,N/mm^2}$$

$Z = I/y$.

$$Z = \frac{\pi/64\, d^4}{d/2}$$

$$\boxed{Z = \pi/32\, d^3}$$

$$Z = \pi/32 \times (50)^3$$

$$\boxed{Z = 12273\,mm^3}$$

Shear stress due to bending moment,

$$\tau = \frac{T\times16}{\pi d^3}$$

$$\tau = \frac{16T}{\pi(50)^3} \Rightarrow \boxed{\tau = 0.0407\times10^{-3}\,T\,N/mm^2}$$

$$\frac{T}{J} = \frac{\tau}{r}$$

$$\frac{T}{\pi/32\, d^4} = \frac{\tau}{d/2}$$

$$\boxed{\tau = \frac{T\times16}{\pi d^3}}$$

Max. principle stress

$$\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x+\sigma_y)^2+4\tau^2}.$$

$$\sigma_1 = \frac{163}{2} + \frac{1}{2}\left[\sqrt{(163)^2+4(0.0407\times10^{-3}T)^2}\right.$$

$$\boxed{\sigma_1 = 81.5+\sqrt{6642.5+1.65\times10^{-9}T^2}\ N/mm^2}$$

Minimum principle stress

$$\boxed{\sigma_2 = 81.5-\sqrt{6642.5+1.65\times10^{-9}T^2}\ N/mm^2}$$

Max. shear stress

$$\tau_{max} : \frac{1}{2}\left[\sqrt{(\sigma_1)^2+4\tau^2}\right]$$

$$= \frac{1}{2}\left[\sqrt{(163)^2+4(0.0407\times10^{-3}T^2)}\right.$$

$$\boxed{\tau_{max} = \sqrt{6642.5+1.65\times10^{-9}T^2}\ N/mm^2}$$

According to maximum principle stress theory.

$$\sigma_1 = \sigma_y.$$

$$81.5+\sqrt{6642.5+1.65\times10^{-9}T^2} = 200.$$

$$6642.5+1.65\times10^{-9}T^2 = (200-81.5)^2 = 14042.$$

$$T^2 = \frac{14042 - 6642.5}{1.65\times10^{-9}} = 4485\times10^{9}$$

$\boxed{T = 2118\times10^{3}\ \text{N-mm}}$ (or) $\boxed{T = 2118\ \text{N-m.}}$

2). According to max. shear stress theory.

$$\sigma_1 - \sigma_2 = \sigma_y.$$

$$\left(81.5 + \sqrt{6642.5 + 1.65\times10^{-9}T^2}\right) - \left(81.5 - \sqrt{6642.5 + 1.65\times10^{-9}T^2}\right)$$

$$= 200.$$

$$2\left(\sqrt{6642.5 + 1.05\times10^{-9}T^2}\right) = 200$$

$$\sqrt{6642.5 + 1.65\times10^{-9}\,T^2} = 100.$$

Square on both sides.

$$6642.5 + 1.65\times10^{-9}\ T^2 = (100)^2.$$

$$T^2 = \frac{(100)^2 - 6642.5}{1.65\times10^{-9}}.$$

$$\boxed{T = 1426.48\times10^{3}\ \text{N-mm.}}$$

$$\boxed{T = 1426.48\ \text{N-m.}}$$

3). According to maximum distortion strain energy theory.

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_3\sigma_1 = \sigma_y^2.$$

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma_y^2$$

$$\left[81.5+\sqrt{6642.5+1.65\times10^{-9}T^2}\right]^2 + \left[81.5-\sqrt{6642.5+1.65\times10^{-9}T^2}\right]^2$$

$$-\left[81.5+\sqrt{6642.5+1.65\times10^{-9}T^2}\right]\left[81.5-\sqrt{6642.5+1.65\times10^{-9}T^2}\right]$$

$$=[200]^2.$$

$$2\left[(81.5)^2+6642.5+1.65\times10^{-9}T^2\right]-\left[(81.5)^2-6642.5-1.65\times10^{-9}T^2\right]=(200)^2.$$

$$(81.5)^2 + 3\times 6642.5 + 1.65\times10^{-9}T^2 = (200)^2.$$

$$26570 + 4.95\times10^{-9}T^2 = 40000$$

$$T^2 = \frac{40000-26570}{4.95\times10^{-9}} = 2713\times10^9.$$

$$\boxed{T = 1647\times10^3 \text{ N-mm.}}$$

$$\boxed{T = 1647 \text{ N-m.}}$$