CHAPTER 14

OSCILLATIONS

14.1 PERIODIC MOTION

1. *What is periodic motion ? Give some of its examples.*

Periodic motion. *Any motion that repeats itself over and over again at regular intervals of time is called periodic or harmonic motion.* The smallest interval of time after which the motion is repeated is called its time period. The time period is denoted by T and its SI unit is second.

Examples of periodic motion :

(*i*) The motion of any planet around the sun in an elliptical orbit is periodic. The period of revolution of Mercury is 87.97 days.

(*ii*) The motion of the moon around the earth is periodic. Its time period is 27.3 days.

(*iii*) The motion of Halley's comet around the sun is periodic. It appears on the earth after every 76 years.

(*iv*) The motion of the hands of a clock is periodic.

(*v*) The heart beats of a human being are periodic. The periodic time is about 0.8 second for a normal person.

14.2 OSCILLATORY OR HARMONIC MOTION

2. *What is oscillatory motion ? Give some of its examples.*

Oscillatory motion. *If a body moves back and forth repeatedly about its mean position, its motion is said to be oscillatory or vibratory or harmonic motion.* Such a motion repeats itself over and over again about a mean position such that it remains confined within well defined limits (known as extreme positions) on either side of the mean position.

Examples of oscillatory motion :

(*i*) The swinging motion of the pendulum of a wall clock.

(*ii*) The oscillations of a mass suspended from a spring.

(*iii*) The motion of the piston of an automobile engine.

(*iv*) The vibrations of the string of a guitar.

(*v*) When a freely suspended bar magnet is displaced from its equilibrium position along north- south line and released, it executes oscillatory motion.

14.3 PERIODIC MOTION VS. OSCILLATORY MOTION

3. *Every oscillatory motion is necessarily periodic but every periodic motion need not be oscillatory. Justify.*

Distinction between periodic and oscillatory motions. Every oscillatory motion is necessarily periodic because it is repeated at regular intervals of time. In addition, it is bounded about one mean position. But every periodic motion need not be oscillatory. For example, the earth completes one revolution around the sun in 1 year but it is not a to and fro motion about any mean position. Hence its motion is periodic but not oscillatory.

14.5 PERIODIC, HARMONIC AND NON-HARMONIC FUNCTIONS

5. *Distinguish between periodic, harmonic and non-harmonic functions. Give examples of each.*

Periodic, harmonic and non-harmonic functions. *Any function that repeats itself at regular intervals of its argument is called a periodic function.* The following sine and cosine functions are periodic with period T:

$$f(t) = \sin \omega t = \sin \frac{2\pi t}{T}$$

and $$g(t) = \cos \omega t = \cos \frac{2\pi t}{T}$$

Figure 14.1. shows how these functions vary with time t.

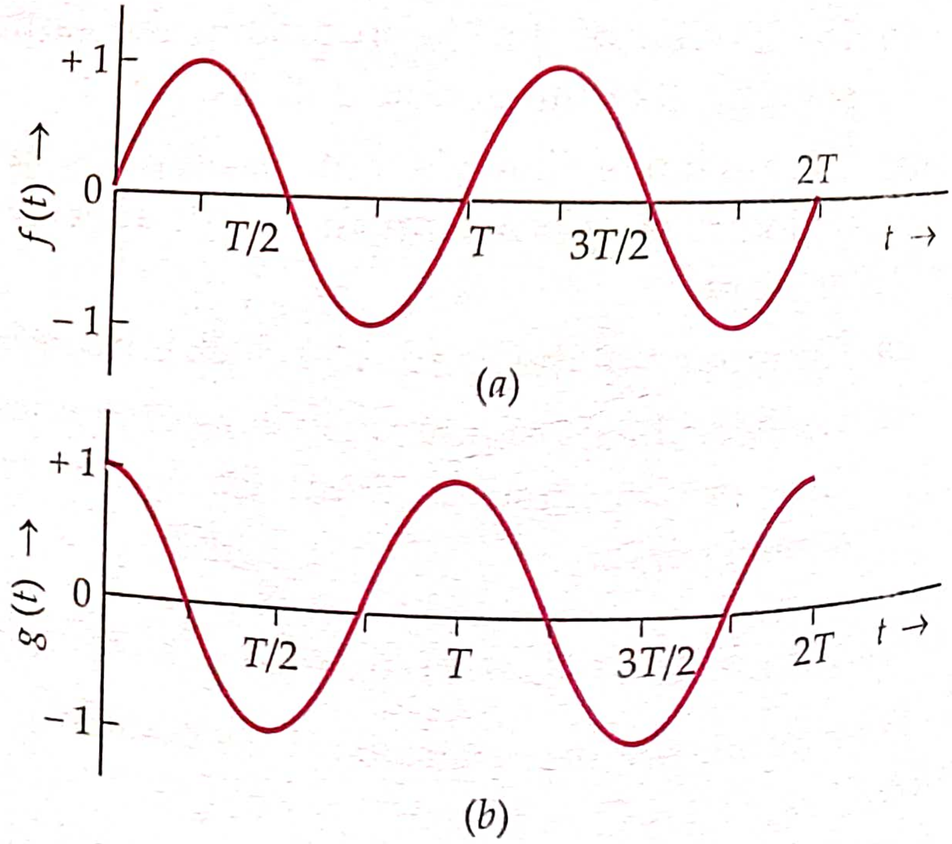

Fig. 14.1 Periodic functions which are harmonic.

Obviously, these functions vary between a maximum value +1 and minimum value −1 passing through zero in between. *The periodic functions which can be represented by a sine or cosine curve are called* **harmonic functions.**

All harmonic functions are necessarily periodic but all periodic functions are not harmonic. The periodic functions which cannot be represented by single sine or cosine function are called **non-harmonic functions.** Fig. 14.2 shows some periodic functions which repeat themselves in a period T but are not harmonic.

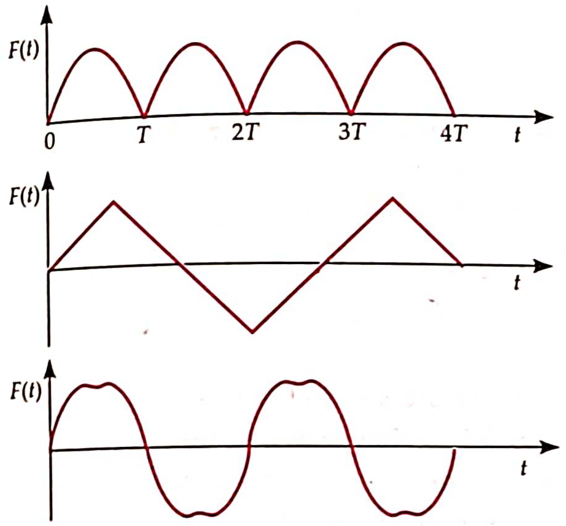

Fig. 14.2 Some non-harmonic periodic functions.

Any non-harmonic periodic function can be constructed from two or more harmonic functions.

One such function is : $F(t) = a_1 \sin \omega t + a_2 \sin 2\omega t$

It can be easily checked that the functions $\tan \omega t$ and $\cot \omega t$ are periodic with period $T = \pi/\omega$ while $\sec \omega t$ and $\operatorname{cosec} \omega t$ are periodic with period $T = 2\pi/\omega$. Thus

$$\tan\left\{\omega\left(t+\frac{\pi}{\omega}\right)\right\} = \tan(\omega t + \pi) = \tan \omega t$$

$$\sec\left\{\omega\left(t+\frac{2\pi}{\omega}\right)\right\} = \sec(\omega t + 2\pi) = \sec \omega t$$

But such functions take values between zero and infinity. So these functions cannot be used to represent displacement functions in periodic motions because displacement always takes a finite value in any physical situation.

Examples Based on Periodic and Harmonic Functions

Concepts Used

1. A function which can be represented by a single sine or cosine function is a harmonic function otherwise non-harmonic.
2. A periodic function can be expressed as the sum of sine and cosine functions of different time periods with suitable coefficients.

Example 1. *On an average a human heart is found to beat 75 times in a minute. Calculate its beat frequency and period.* [NCERT]

Solution. Beat frequency of the heart,

$$\nu = \frac{\text{No. of beats}}{\text{Time taken}} = \frac{75}{1\text{ min}}$$

$$= \frac{75}{60\text{ s}} = 1.25\text{ s}^{-1} = \mathbf{1.25\ Hz}$$

Beat period, $T = \frac{1}{\nu} = \frac{1}{1.25\text{ s}^{-1}} = \mathbf{0.8\ s.}$

Example 2. *Which of the following functions of time represent (a) periodic and (b) non-periodic motion ? Give the period for each case of periodic motion. [ω is any positive constant].* [NCERT]

(i) $\sin \omega t + \cos \omega t$ (ii) $\sin \omega t + \cos 2\omega t + \sin 4\omega t$

(iii) $e^{-\omega t}$ (iv) $\log(\omega t)$.

Solution. (i) Here $x(t) = \sin \omega t + \cos \omega t$

$$= \sqrt{2}\left[\sin \omega t \cos\frac{\pi}{4} + \cos \omega t \sin\frac{\pi}{4}\right]$$

$$= \sqrt{2}\sin(\omega t + \pi/4)$$

Moreover,

$$x\left(t+\frac{2\pi}{\omega}\right) = \sqrt{2}\sin[\omega(t+2\pi/\omega)+\pi/4]$$

$$= \sqrt{2}\sin\left(\omega t + 2\pi + \frac{\pi}{4}\right)$$

$$= \sqrt{2}\sin\left(\omega t + \frac{\pi}{4}\right) = x(t)$$

Hence $\sin \omega t + \cos \omega t$ is a **periodic function** with time period equal to $2\pi/\omega$.

(ii) Here $x(t) = \sin \omega t + \cos 2\omega t + \sin 4\omega t$

$\sin \omega t$ is a periodic function with period

$$= 2\pi/\omega = T$$

$\cos 2\omega t$ is a periodic function with period

$$= 2\pi/2\omega = \pi/\omega = T/2$$

$\sin 4\omega t$ is a periodic function with period

$$= 2\pi/4\omega = \pi/2\omega = T/4$$

Clearly, the entire function $x(t)$ repeats after a minimum time $T = 2\pi/\omega$. Hence the given function is **periodic.**

(iii) The function $e^{-\omega t}$ decreases monotonically to zero as $t \to \infty$. It is an exponential function with a negative exponent of e, where $e \simeq 2.71828$. It never repeats its value. So it is **non-periodic.**

(iv) The function $\log(\omega t)$ increases monotonically with time. As $t \to \infty$, $\log(\omega t) \to \infty$. It never repeats its value. So it is **non-periodic.**

Problems For Practice

Which of the following functions of time represent (a) simple harmonic motion, (b) periodic but not simple harmonic and (c) non-periodic motion ? Find the period of each periodic motion. Here ω is a positive real constant.

1. $\sin \omega t + \cos \omega t$. (Ans. Simple harmonic)
2. $\sin \pi t + 2\cos 2\pi t + 3\sin 3\pi t$. (Ans. Periodic but not simple harmonic)
3. $\cos(2\omega t + \pi/3)$. (Ans. Simple harmonic)
4. $\sin^2 \omega t$. (Ans. Periodic but not simple harmonic)
5. $\cos \omega t + 2\sin^2 \omega t$. (Ans. Periodic but not simple harmonic)

HINTS

1. $\sin \omega t + \cos \omega t = \sqrt{2}\sin(\omega t + \pi/4)$, $T = 2\pi/\omega$.
2. Each term represents S.H.M.

   Period of $\sin \pi t$, $T = \dfrac{2\pi}{\pi} = 2\text{s}$

   Period of $2\cos 2\pi t = \dfrac{2\pi}{2\pi} = 1\text{s} = T/2$

   Period of $3\sin 3\pi t = \dfrac{2\pi}{3\pi} = \dfrac{2}{3}\text{s} = T/3$

   The sum is not simple harmonic but periodic with $T = 2$ s.
3. $\cos(2\omega t + \pi/3)$ represents S.H.M. with
   $$T = 2\pi/2\omega = \pi/\omega.$$
4. $\sin^2 \omega t = 1/2 - (1/2)\cos 2\omega t$.

   The function does not represent S.H.M. but is periodic with $T = 2\pi/2\omega = \pi/\omega$.
5. $\cos \omega t + 2\sin^2 \omega t = \cos \omega t + 1 - \cos 2\omega t$
   $$= 1 + \cos \omega t - \cos 2\omega t$$
   $\cos \omega t$ represents S.H.M. with $T = 2\pi/\omega$

   $\cos 2\omega t$ represents S.H.M. with period
   $$= 2\pi/2\omega = \pi/\omega = T/2$$
   The combined function does not represent S.H.M. but is periodic with $T = 2\pi/\omega$.

14.6 SIMPLE HARMONIC MOTION

6. *What is meant by simple harmonic motion ? Give some examples.*

Simple harmonic motion. *A particle is said to execute simple harmonic motion if it moves to and fro about a mean position under the action of a restoring force which is directly proportional to its displacement from the mean position and is always directed towards the mean position.*

If the displacement of the oscillating body from the mean position is small, then

Restoring force ∝ Displacement

$$F \propto x \quad \text{or} \quad F = -kx$$

This equation defines S.H.M. Here k is a positive constant called **force constant** or **spring factor** and is *defined as the restoring force produced per unit displacement*. The SI unit of k is Nm^{-1}. The negative sign in the above equation shows that the restoring force F always acts in the opposite direction of the displacement x.

Now, according to Newton's second law of motion,

$$F = ma$$

$$\therefore \quad ma = -kx$$

$$\text{or} \quad a = -\frac{k}{m}x \quad \text{i.e.,} \quad a \propto x$$

Hence simple harmonic motion may also be defined as follows :

A particle is said to possess simple harmonic motion if it moves to and fro about a mean position under an acceleration which is directly proportional to its displacement from the mean position and is always directed towards that position.

Examples of simple harmonic motion :

(*i*) Oscillations of a loaded spring.

(*ii*) Vibrations of a tuning fork.

(*iii*) Vibrations of the balance wheel of a watch.

(*iv*) Oscillations of a freely suspended magnet in a uniform magnetic field.

7. *State some important features of simple harmonic motion.*

Some important features of S.H.M. :

(*i*) The motion of the particle is periodic.

(*ii*) It is the oscillatory motion of simplest kind in which the particle oscillates back and forth about its mean position with constant amplitude and fixed frequency.

(*iii*) Restoring force acting on the particle is proportional to its displacement from the mean position.

(*iv*) The force acting on the particle always opposes the increase in its displacement.

(*v*) A simple harmonic motion can always be expressed in terms of a single harmonic function of sine or cosine.

14.7 DIFFERENTIAL EQUATION FOR S.H.M.

8. *Write down the differential equation for S.H.M. Give its solution. Hence obtain expression for time period of S.H.M.*

Differential equation of S.H.M. In S.H.M., the restoring force acting on the particle is proportional to its displacement. Thus

$$F = -kx$$

The negative sign shows that F and x are oppositely directed. Here k is spring factor or force constant.

By Newton's second law,

$$F = m\frac{d^2x}{dt^2}$$

where m is the mass of the particle and $\frac{d^2x}{dt^2}$ is its acceleration.

$$\therefore \quad m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Put $\frac{k}{m} = \omega^2$, then $\frac{d^2x}{dt^2} = -\omega^2 x$

or
$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \qquad ...(1)$$

This is the **differential equation of S.H.M.** Here ω is the angular frequency. Clearly, x should be such a function whose second derivative is equal to the function itself multiplied with a negative constant. So a possible solution of equation (1) may be of the form

$$x = A\cos(\omega t + \phi_0)$$

Then
$$\frac{dx}{dt} = -\omega A \sin(\omega t + \phi_0)$$

and
$$\frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t + \phi_0) = -\omega^2 x$$

or
$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

which is same as equation (1). Hence the solution of the equation (1) is

$$x = A\cos(\omega t + \phi_0) \qquad ...(2)$$

It gives displacement of the harmonic oscillator at any instant t.

Here A is the *amplitude* of the oscillating particle.

$\phi = \omega t + \phi_0$, is the phase of the oscillating particle.

ϕ_0 is the initial *phase* (at $t = 0$) or epoch.

Time period of S.H.M. If we replace t by $t + \frac{2\pi}{\omega}$ in equation (2), we get

$$x = A\cos\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi_0\right]$$

$$= A\cos(\omega t + 2\pi + \phi_0) = A\cos(\omega t + \phi_0)$$

i.e., the motion repeats after time interval $\frac{2\pi}{\omega}$. Hence $\frac{2\pi}{\omega}$ is the time period of S.H.M.

$$\therefore \quad T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} \qquad \left[\because \omega^2 = \frac{k}{m}\right]$$

or
$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{\text{Inertia factor}}{\text{Spring factor}}}$$

In general, m is called inertia factor and k the spring factor.

14.8 SOME IMPORTANT TERMS CONNECTED WITH S.H.M.

9. *Define the terms harmonic oscillator, displacement, amplitude, cycle, time period, frequency, angular frequency, phase and epoch with reference to oscillatory motion.*

Some important terms connected with S.H.M :

(*i*) **Harmonic oscillator.** *A particle executing simple harmonic motion is called harmonic oscillator.*

(*ii*) **Displacement.** *The distance of the oscillating particle from its mean position at any instant is called its displacement.* It is denoted by x.

There can be other kind of displacement variables. These can be voltage variations in time across a capacitor in an a.c. circuit, pressure variations in time in the propagation of a sound wave, the changing electric and magnetic fields in the propagation of a light wave, etc.

(*iii*) **Amplitude.** *The maximum displacement of the oscillating particle on either side of its mean position is called its amplitude.* It is denoted by A. Thus $x_{max} = \pm A$.

(*iv*) **Oscillation or cycle.** *One complete back and forth motion of a particle starting and ending at the same point is called a cycle or oscillation or vibration.*

(*v*) **Time period.** *The time taken by a particle to complete one oscillation is called its time period. Or, it is the smallest time interval after which the oscillatory motion repeats.* It is denoted by T.

(*vi*) **Frequency.** *It is defined as the number of oscillations completed per unit time by a particle.* It is denoted by ν (nu). Frequency is equal to the reciprocal of time period. That is,

$$\nu = \frac{1}{T}$$

Clearly, the unit of frequency is (second)$^{-1}$ or s^{-1}. It is also expressed as *cycles per second* (cps) or *hertz* (Hz).

SI unit of frequency = s^{-1} = cps = Hz.

(*vii*) **Angular frequency.** *It is the quantity obtained by multiplying frequency ν by a factor of 2π.* It is denoted by ω.

Thus, $\omega = 2\pi\nu = \frac{2\pi}{T}$

SI unit of angular frequency = rad s^{-1}.

(*viii*) **Phase.** *The phase of a vibrating particle at any instant gives the state of the particle as regards its position and the direction of motion at that instant.* It is equal to the argument of sine or cosine function occurring in the displacement equation of the S.H.M. Suppose a simple harmonic equation is represented by

$$x = A\cos(\omega t + \phi_0)$$

Then phase of the particle is : $\phi = \omega t + \phi_0$

Clearly, the phase ϕ is a function of time t. It is usually expressed either as the fraction of the time period T or fraction of angle 2π that has elapsed since the vibrating particle last passed its mean position in the positive direction.

$\phi = \omega t + \phi_0$	0	$\pi/2$	π	$3\pi/2$	2π
$x = A\cos(\omega t + \phi_0)$	$+A$	0	$-A$	0	$+A$

Thus the phase ϕ gives an idea about the position and the direction of motion of the oscillating particle.

(*ix*) **Initial phase or epoch.** *The phase of a vibrating particle corresponding to time $t = 0$ is called initial phase or epoch.*

At $t = 0$, $\quad \phi = \phi_0$

The constant ϕ_0 is called initial phase or epoch. It tells about the initial state of motion of the vibrating particle.

14.12 PHASE RELATIONSHIP BETWEEN DISPLACEMENT, VELOCITY AND ACCELERATION

14. *Draw displacement-time, velocity-time and acceleration-time graphs for a particle executing simple harmonic motion. Discuss their phase relationship.*

Inter-relationship between particle displacement, velocity and acceleration in S.H.M. If a particle executing S.H.M. passes through its positive extreme position ($x = + A$) at time $t = 0$, then its displacement equation can be written as

$$x(t) = A \cos \omega t$$

Velocity, $$v(t) = \frac{dx}{dt} = -\omega A \sin \omega t$$

$$= \omega A \cos\left(\omega t + \frac{\pi}{2}\right)$$

Acceleration, $$a(t) = \frac{dv}{dt} = -\omega^2 A \cos \omega t$$

$$= \omega^2 A \cos(\omega t + \pi)$$

Using the above relations, we determine the values of displacement, velocity and acceleration at various instants t for one complete cycle as illustrated below.

Time, t	0	$\frac{T}{4}$	$\frac{T}{2}$	$\frac{3T}{4}$	T
Phase angle, $\omega t = \frac{2\pi}{T} t$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Displacement, $x(t)$	$+ A$ max.	0 min.	$- A$ max.	0 min.	$+ A$ max.
Velocity, $v(t)$	0 min.	$- \omega A$ max.	0 min.	$+ \omega A$ max.	0 min.
Acceleration, $a(t)$	$- \omega^2 A$ max.	0 min.	$+ \omega^2 A$ max.	0 min.	$- \omega^2 A$ max.

In Fig. 14.8, we have plotted separately the x versus t, v versus t and a versus t curves for a simple harmonic motion.

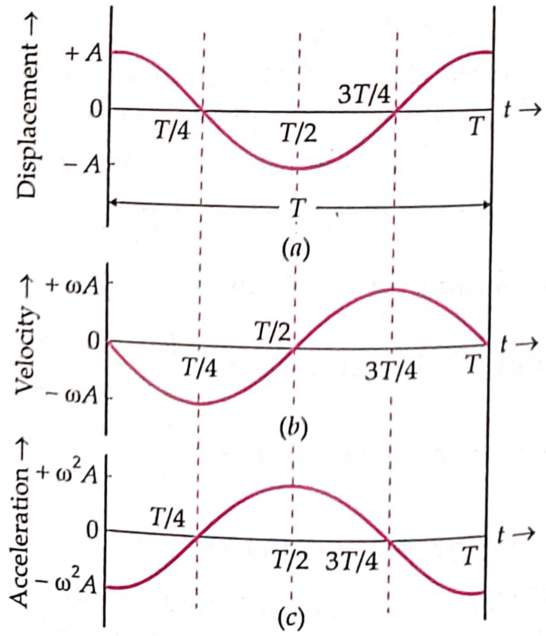

Fig. 14.8 Relation between velocity, displacement and acceleration in S.H.M.

12. Here $T = 4\,s$, $A = 2\,cm$,

$$\omega = \frac{2\pi}{T} = \frac{2 \times 3.14}{4} = 1.57 \text{ rad s}^{-1}$$

$$v_{max} = \omega A = 1.57 \times 2 = \mathbf{3.14 \text{ cm s}^{-1}}.$$

At $y = A/2 = 1\,cm$,

$$v = \omega\sqrt{A^2 - y^2} = 1.57\sqrt{2^2 - 1^2} = \mathbf{2.72 \text{ cm s}^{-1}}.$$

At the turning points, acceleration is maximum

$\therefore$ $a_{max} = \omega^2 A = (1.57)^2 \times 2 = \mathbf{4.93 \text{ cm s}^{-2}}$.

At $y = 0.75\,cm$,

$$a = \omega^2 y = (1.57)^2 \times 0.75 = \mathbf{1.85 \text{ cm s}^{-2}}.$$

13. Here $y = \sqrt{3}/2\,A$

$\therefore$ $v = \omega\sqrt{A^2 - y^2} = \omega\sqrt{A^2 - 3/4\,A^2}$

$$= \frac{1}{2}\omega A = \frac{1}{2}v_{max}.$$

14. Let $y = A \sin \omega t$

Then $v = \dfrac{dy}{dt} = \omega A \cos \omega t = \dfrac{2\pi}{T} A \cos \dfrac{2\pi}{T} t$

$\therefore$ $3.142 = \dfrac{2 \times 3.142}{12} A \cos \dfrac{2\pi}{12} \times 2$

or $A = \mathbf{12 \text{ cm}}$ and length of path $= 2A = \mathbf{24 \text{ cm}}$.

15. $a_{max} = \omega^2 A = \mu g$

$\therefore$ $A = \dfrac{\mu g}{\omega^2} = \dfrac{\mu g T^2}{4\pi^2}$

$$= \frac{0.4 \times 9.8 \times (1)^2}{4 \times 10} = 0.098 \text{ m} = \mathbf{9.8 \text{ cm}}.$$

16. Take $a_{max} = \omega^2 A = \left(\dfrac{2\pi}{T}\right)^2 A = g$.

17. Length of stroke $= 2A = 10$ cm.

18. Here $m = 50$ kg, $\nu = 2$ Hz,

$A = 5\text{ cm} = 0.05\text{ m} = 4 \times 9.87 \times 2^2 \times 0.05 = 7.9 \text{ ms}^{-2}$.

$a_{max} = \omega^2 A = 4\pi^2 \nu^2 A$

Max. force on the man

$= m(g + a_{max}) = 50(10 + 7.9) = 895.0\text{ N} = \mathbf{89.5 \text{ kg f}}$

Min. force on the man

$= m(g - a_{max}) = 50(10 - 7.9) = 105.0\text{ N} = \mathbf{10.5 \text{ kg f}}$.

14.13 ENERGY IN S.H.M. : KINETIC AND POTENTIAL ENERGIES

15. *Derive expressions for the kinetic and potential energies of a simple harmonic oscillator. Hence show that the total energy is conserved in S.H.M. In which positions of the oscillator, is the energy wholly kinetic or wholly potential ?*

Total energy in S.H.M. The energy of a harmonic oscillator is partly kinetic and partly potential. When a body is displaced from its equilibrium position by doing work upon it, it acquires potential energy. When the body is released, it begins to move back with a velocity, thus acquiring kinetic energy.

(*i*) **Kinetic energy.** At any instant, the displacement of a particle executing S.H.M. is given by

$$x = A\cos(\omega t + \phi_0)$$

$\therefore$ Velocity, $v = \dfrac{dx}{dt} = -\omega A \sin(\omega t + \phi_0)$

Hence kinetic energy of the particle at a displacement x is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi_0)$$

But $A^2 \sin^2(\omega t + \phi_0) = A^2[1 - \cos^2(\omega t + \phi_0)]$

$$= A^2 - A^2\cos^2(\omega t + \phi_0) = A^2 - x^2$$

$\therefore$
$$K = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi_0)$$

or
$$K = \frac{1}{2}m\omega^2(A^2 - x^2) = \frac{1}{2}k(A^2 - x^2)$$

(*ii*) **Potential energy.** When the displacement of a particle from its equilibrium position is x, the restoring force acting on it is

$$F = -kx$$

If we displace the particle further through a small distance dx, then work done against the restoring force is given by

$$dW = -F\,dx = +k\,x\,dx$$

The total work done in moving the particle from mean position ($x = 0$) to displacement x is given by

$$W = \int dW = \int_0^x k\,x\,dx = k\left[\frac{x^2}{2}\right]_0^x = \frac{1}{2}kx^2$$

This work done against the restoring force is stored as the potential energy of the particle. Hence potential energy of a particle at displacement x is given by

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi_0)$$

(*iii*) **Total energy.** At any displacement x, the total energy of a harmonic oscillatory is given by

$$E = K + U = \frac{1}{2}k(A^2 - x^2) + \frac{1}{2}kx^2$$

or
$$E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2A^2 = 2\pi^2 m\nu^2 A^2$$

$[\because \omega = 2\pi\nu]$

Thus the total mechanical energy of a harmonic oscillator is independent of time or displacement. Hence *in the absence of any frictional force, the total energy of a harmonic oscillator is conserved.*

Obviously, the total energy of particle in S.H.M. is

(i) directly proportional to the mass m of the particle,

(ii) directly proportional to the square of its frequency ν, and

(iii) directly proportional to the square of its vibrational amplitude A.

Graphical representation. *At the mean position*, $x = 0$

Kinetic energy, $K = \frac{1}{2} k (A^2 - 0^2) = \frac{1}{2} kA^2$

Potential energy, $U = \frac{1}{2} k (0^2) = 0$

Hence at the mean position, the energy is all kinetic.

At the extreme positions, $x = \pm A$

Kinetic energy, $K = \frac{1}{2} k (A^2 - A^2) = 0$

Potential energy, $U = \frac{1}{2} kA^2$

Hence at the two extreme positions, the energy is all potential.

Figure 14.12 shows the variations of kinetic energy K, potential energy U and total energy E with displacement x. The graphs for K and U are parabolic while that for E is a straight line parallel to the displacement axis. At $x = 0$, the energy is all kinetic and for $x = \pm A$, the energy is all potential.

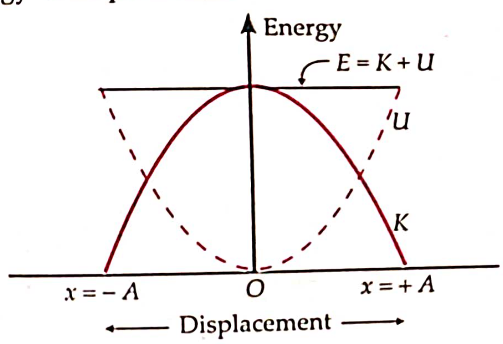

Fig. 14.12 K, U and E as functions of displacement x for a harmonic oscillator.

Figure 14.13 shows the variations of energies K, U and E of a harmonic oscillator with time t. Clearly, twice in each cycle, both kinetic and potential energies assume their peak values. Both of these energies are periodic functions of time, the time period of each being $T/2$.

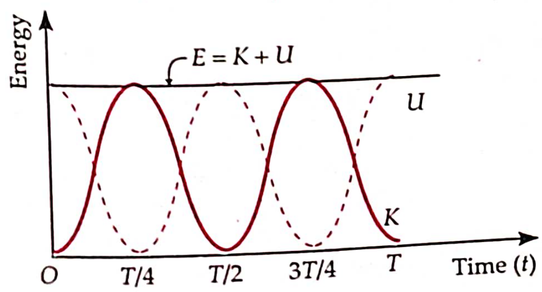

Fig. 14.13 K, U and E as functions of time t for a harmonic oscillator.

Examples based on Energy of SHM

Formulae Used

1. P.E. at displacement y from the mean position,

$$E_p = \frac{1}{2} ky^2 = \frac{1}{2} m\omega^2 y^2 = \frac{1}{2} m\omega^2 A^2 \sin^2 \omega t$$

2. K.E. at displacement y from the mean position,

$$E_k = \frac{1}{2} k (A^2 - y^2) = \frac{1}{2} m\omega^2 (A^2 - y^2)$$

$$= \frac{1}{2} m\omega^2 A^2 \cos^2 \omega t$$

3. Total energy at any point,

$$E = \frac{1}{2} kA^2 = \frac{1}{2} m\omega^2 A^2 = 2\pi^2 m A^2 \nu^2$$

Units Used

Energies E_p, E_k and E are in joule, displacement in metre, force constant k in Nm^{-1} and angular frequency ω in rad s^{-1}.

Example 23. *A block whose mass is 1 kg is fastened to a spring. The spring has a spring constant of 50 N m^{-1}. The block is pulled to a distance $x = 10$ cm from its equilibrium position at $x = 0$ on a frictionless surface from rest at $t = 0$. Calculate the kinetic, potential and total energies of the block when it is 5 cm away from the mean position.* **[NCERT ; Delhi 18]**

Solution. Here $m = 1$ kg, $k = 50$ N m^{-1},

$A = 10$ cm $= 0.10$ m, $y = 5$ cm $= 0.05$ m

Kinetic energy,

$$E_k = \frac{1}{2} k (A^2 - y^2) = \frac{1}{2} \times 50 [(0.10)^2 - (0.05)^2]$$

$$= \mathbf{0.1875\ J}$$

Potential energy,

$$E_p = \frac{1}{2} ky^2 = \frac{1}{2} \times 50 \times (0.05)^2 = \mathbf{0.0625\ J}$$

Total energy,

$$E = E_k + E_p = 0.1875 + 0.0625 = \mathbf{0.25\ J}.$$

Example 24. *A body executes SHM of time period 8 s. If its mass be 0.1 kg, its velocity 1 second after it passes through its mean position be 4 ms^{-1}, find its (i) kinetic energy (ii) potential energy and (iii) total energy.*

Solution. Here $m = 0.1$ kg, $T = 8$ s

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4} \text{ rad s}^{-1}$$

When $t = 1$ s, $v = 4$ ms^{-1}

But $v = \omega A \cos \omega t$

∴ $4 = \frac{\pi}{4} \times A \cos\left(\frac{\pi}{4} \times 1\right) = \frac{\pi}{4} \times A \times \frac{1}{\sqrt{2}}$

or $A = \frac{16\sqrt{2}}{\pi}$ m

Now $y = A \sin \omega t = A \sin \frac{2\pi}{T} t$

$\therefore \quad \frac{A}{\sqrt{2}} = A \sin \frac{2\pi}{8} t$

or $\sin \frac{\pi t}{4} = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$

or $\frac{\pi t}{4} = \frac{\pi}{4}$ or $t = 1$ s.

4. When $t = \frac{\pi}{4}$ s, $y = 0.08\sqrt{2}$ m

As $y = A \sin \omega t = A \sin \frac{2\pi}{T} t$

$\therefore \quad 0.08\sqrt{2} = A \sin \frac{2\pi}{2\pi} \times \frac{\pi}{4} = A \sin \frac{\pi}{4}$

or $0.08\sqrt{2} = A \times \frac{1}{\sqrt{2}}$

$\therefore \quad A = 0.08\sqrt{2} \times \sqrt{2} = \mathbf{0.16\ m.}$

Total energy $= \frac{1}{2} m \omega^2 A^2 = \frac{1}{2} m \left(\frac{2\pi}{T}\right)^2 A^2$

$\therefore \quad 1.024 \times 10^{-3} = \frac{1}{2} m \left(\frac{2\pi}{2\pi}\right)^2 \times (0.16)^2$

or $m = \frac{2 \times 1.024 \times 10^{-3}}{(0.16)^2} = \mathbf{0.08\ kg.}$

5. Here $\nu = 1/\pi$ Hz, $E = 10$ J, $v_{max} = 0.4$ ms^{-1}

Now $v_{max} = \omega A = 2\pi\nu A$

$\therefore \quad A = \frac{v_{max}}{2\pi\nu} = \frac{0.4 \times \pi}{2\pi \times 1} = 0.2$ m

As $E = \frac{1}{2} kA^2$

$\therefore \quad k = \frac{2E}{A^2} = \frac{2 \times 10}{(0.2)^2} = \mathbf{500\ Nm^{-1}}$

$(E_p)_{max} = E = \mathbf{10\ J.}$

6. Here $F = mg = 1.0 \times 10$ N, $y = 2$ cm $= 0.02$ m

$\therefore \quad k = \frac{F}{y} = \frac{1.0 \times 10}{0.02} = 500$ Nm^{-1}

$T = 2\pi\sqrt{\frac{m}{k}} = 2 \times 3.14 \times \sqrt{\frac{1.0}{500}} = \mathbf{0.28\ s}$

E_k = Work done in pulling the spring through 10 cm or 0.1 m

$= \frac{1}{2} k x^2 = \frac{1}{2} \times 500 \times (0.1)^2 = \mathbf{2.5\ J.}$

14.14 OSCILLATIONS DUE TO A SPRING

16. *Derive an expression for the time-period of the horizontal oscillations of a massless loaded spring.*

Horizontal oscillations of a body on a spring. Consider a massless spring lying on a frictionless horizontal table. Its one end is attached to a rigid support and the other end to a body of mass m. If the body is pulled towards right through a small distance and released, it starts oscillating back and forth about its equilibrium position under the action of the restoring force of elasticity,

$$F = -kx$$

where k is the force constant (restoring force per unit compression or extension) of the spring. The negative sign indicates that the force is directed oppositely to x.

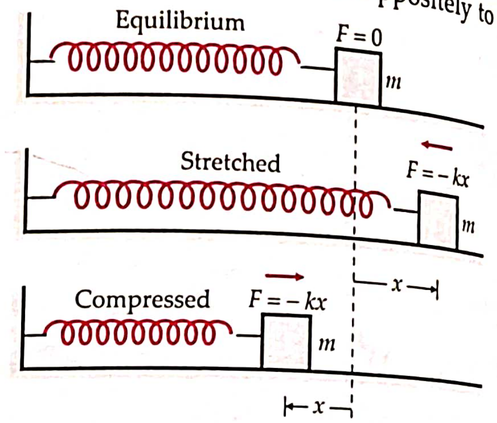

Fig. 14.14 Horizontal oscillations of a loaded spring.

If d^2x/dt^2 is the acceleration of the body, then

$$m\frac{d^2x}{dt^2} = -kx$$

or

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

This shows that the acceleration is proportional to displacement x and acts opposite to it. Hence the body executes simple harmonic motion. Its time period is given by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}}$$

or

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Frequency of oscillation will be

$$\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Clearly, the time period T will be small or frequency ν large if the spring is stiff (high k) and attached body is light (small m).

17. *Deduce an expression for the time-period of the vertical oscillations of a massless loaded spring. Does it depend on acceleration due to gravity ?*

Vertical oscillations of a body on a spring. If a spring is suspended vertically from a rigid support and a body of mass m is attached to its lower end, the

14.16 SIMPLE PENDULUM

19. *Show that for small oscillations the motion of a simple pendulum is simple harmonic. Derive an expression for its time period. Does it depend on the mass of the bob ?*

Simple pendulum. *An ideal simple pendulum consists of a point-mass suspended by a flexible, inelastic and weightless string from a rigid support of infinite mass.* In practice, we can neither have a point-mass nor a weightless string.

In practice, a simple pendulum is obtained by suspending a small metal bob by a long and fine cotton thread from a rigid support.

Expression for time period. In the equilibrium position, the bob of a simple pendulum lies vertically below the point of suspension. If the bob is slightly displaced on either side and released, it begins to oscillate about the mean position.

Suppose at any instant during oscillation, the bob lies at position A when its displacement is $OA = x$ and the thread makes angle θ with the vertical. The forces acting on the bob are

(*i*) Weight mg of the bob acting vertically downwards.

(*ii*) Tension T along the string.

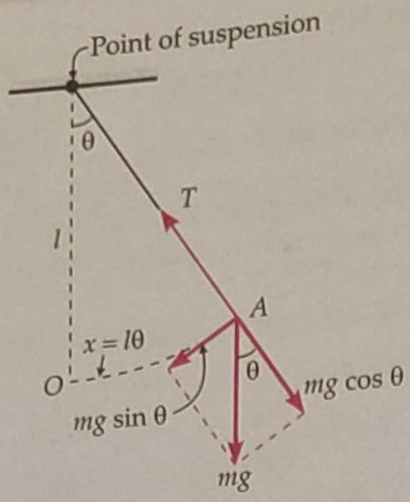

Fig. 14.27 Force acting on the bob of a pendulum.

The force mg has two rectangular components (*i*) the component $mg\cos\theta$ acting along the thread is balanced by the tension T in the thread and (*ii*) the tangential component $mg\sin\theta$ is the net force acting on the bob and tends to bring it back to the mean position. Thus, the restoring force is

$$F = -mg\sin\theta = -mg\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - .....\right)$$

$$= -mg\,\theta\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - .....\right)$$

where θ is in radians. Clearly, oscillations are not simple harmonic because the restoring force F is not proportional to the angular displacement θ.

However, if θ is so small that its higher powers can be neglected, then

$$F = -mg\,\theta$$

If l is the length of the simple pendulum, then

$$\theta\,(\text{rad}) = \frac{\text{arc}}{\text{radius}} = \frac{x}{l}$$

$$\therefore \quad F = -mg\frac{x}{l}$$

or $$ma = -\frac{mg}{l}x$$

or $$a = -\frac{g}{l}x = -\omega^2 x$$

Thus, the acceleration of the bob is proportional to its displacement x and is directed opposite to it. Hence for small oscillations, the motion of the bob is simple harmonic. Its time period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/l}} \quad \text{or} \quad T = 2\pi\sqrt{\frac{l}{g}}$$

Obviously, the time period of a simple pendulum depends on its length l and acceleration due to gravity g but is independent of the mass m of the bob.