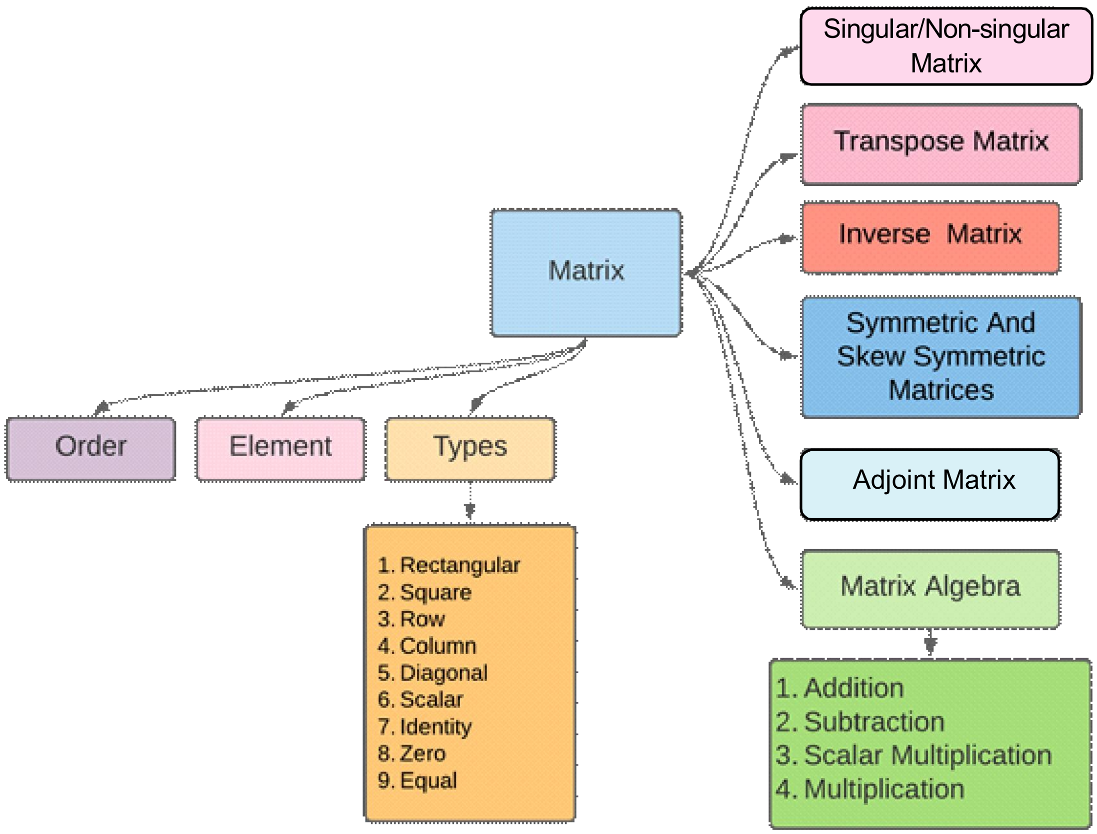
Unit 2

Algebra

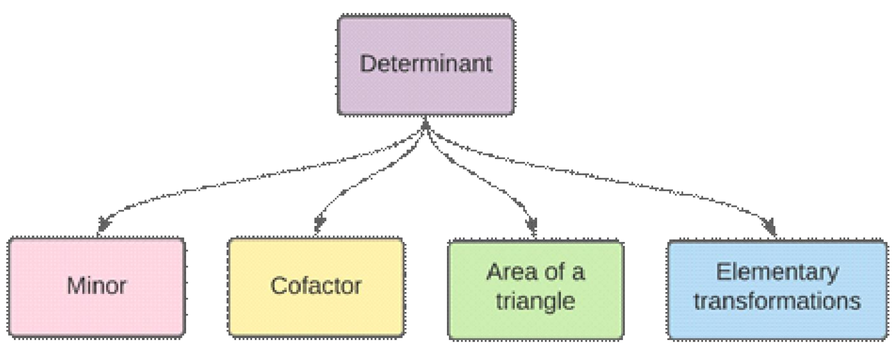
2.0 LEARNING OUTCOMES

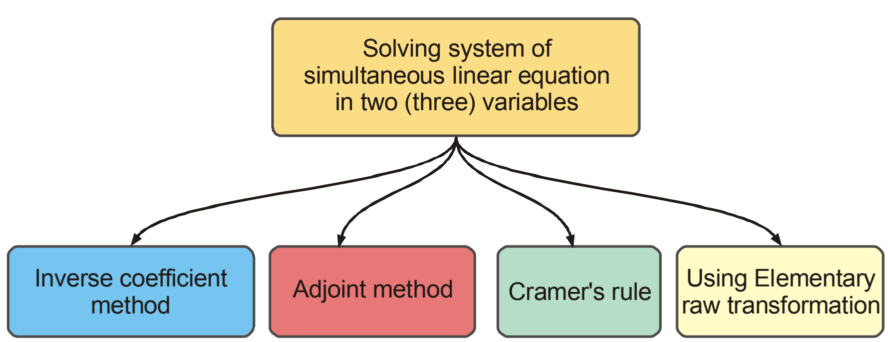
After completion of the unit the students will be able to:

- ❖ Understand and use concept of a matrix and related terms
- ❖ Differentiate between types of matrices
- ❖ Perform mathematical operations:
  - (i) Addition and subtraction of matrices, properties of addition and subtraction of matrices
  - (ii) Scalar multiplication and Multiplication of matrices, properties of multiplication of matrices
- ❖ Understand and use concept of determinant of a matrix.
- ❖ Understand and use properties of determinant
- ❖ Use elementary operations (transformations) to evaluate determinant value
- ❖ Apply concept of determinant to solve problems based on simple applications such as area of triangle and collinearity of the three points.
- ❖ Understand the conditions for inverse matrix to exist and find inverse of a matrix
- ❖ Solve the system of linear equation using:
  - (i) Cramer's rule
  - (ii) Inverse of coefficient matrix method
  - (iii) Adjugate method
  - (iv) Row reduction method
- ❖ Apply concept of matrix and determinant to formulate and solve real- life situations
- ❖ Able to perform simple applications of matrices and determinants including *Leontief input output model* for two variables

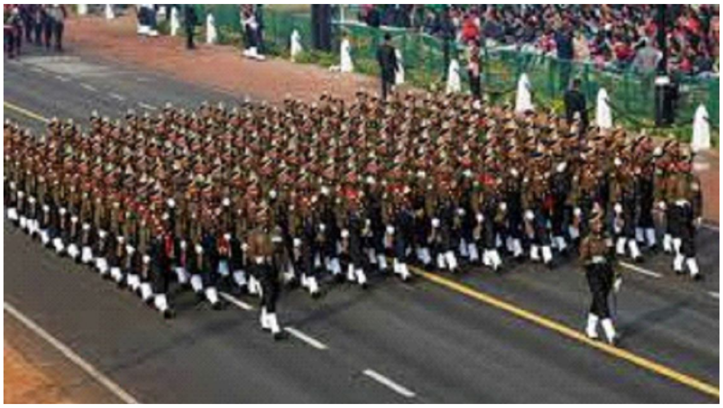


2.1 Concept map

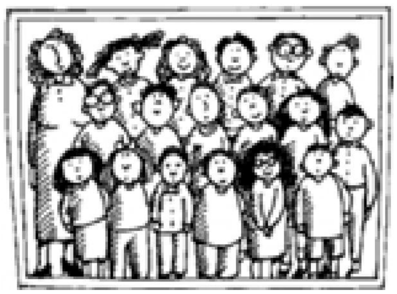




2.2 Introduction



The Republic day parade is a treat for the eyes of every proud Indian. The order and the synchronisation among the army personnel, moving on the beat of drums is a sight to marvel. You can also see the group move together with each personnel holding on to his position in the arrangement of rows and columns.



Another example of elements in a formation is the position and arrangement of students in a class photograph and how easy it is to spot a student standing in the first row and the second column from the top left!

Have you ever had difficulty in finding your seat in a cinema hall? Ever wondered why cinema halls and sports arena have seats arranged in the rows and column formations?

Yes! Among many reasons, it is helpful to locate position of each element in the arrangement of things.

2.3 Matrix

Definition: A rectangular array or table of numbers, symbols, expressions or functions, when arranged in rows and columns, is known as a *matrix* (plural *matrices*). Each member of this arrangement is called an *element* of the matrix.

If we want to write an arrangement of numbers, say 1, 3 and -4

Using matrix, we can write it as $[1 \quad 3 \quad -4]$ or $\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$

Let us now try to arrange x, y, z, w, r and u.

Using matrix, we can write it in so many ways as

$\begin{bmatrix} x & y & z \\ w & r & u \end{bmatrix}$ or $\begin{bmatrix} x & y \\ z & w \\ r & u \end{bmatrix}$ and many more

A matrix is expressed by using capital English alphabet, say A = $[a_{ij}]$ where a_{ij} is the element at the i-th row and j-th column of the matrix and $1 \le i \le m, 1 \le j \le n, \forall i, j \epsilon N$. a_{ij} is also known as general element of matrix A.

For example, In matrix A = $\begin{bmatrix} 2+i & 5 \\ -1 & 2k \\ 4-2i & 0 \end{bmatrix}$

The elements in matrix A are written as:

$a_{11} = 2 + i$
$a_{12} = 5$
$a_{21} = -1,$
$a_{22} = 2k,$
$a_{31} = 4 - 2i$ and $a_{32} = 9$

and

Definition: For a matrix A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots \dots a_{1n}. \\ a_{21} & a_{22} & a_{23} \dots \dots . a_{2n} \\ a_{31} & a_{32} & a_{n3} \dots \dots a_{mn} \end{bmatrix}$ having 'm' number of rows and 'n' number of columns, the expression m x n is called the *order* of the given matrix A where $1 \le i \le m, 1 \le j \le n, \forall i, j \epsilon N$ and is read as 'm by n'.

The elements a_{11}, a_{22}, a_{33}, ... where $a_{ij}, \forall i = j$ are called elements of the *diagonal* a diagonal elements of the matrix

And the elements where $a_{ij}, \forall i \neq j$ are called elements of the non-*diagonal* a non-diagonal elements of the matrix

The order of matrix A = $\begin{bmatrix} 2 & 5 \\ -1 & 2 \\ 4 & 0 \end{bmatrix}$ is 3×2, having 6 elements in total

B = $\begin{bmatrix} x & y & z \\ w & r & u \end{bmatrix}$ is a matrix of order 2×3, having 6 elements

Though the elements of a matrix can take any value, for the scope of this chapter we shall consider only those matrices whose elements are real numbers or functions taking real values

Example 1

Shalabh has 3 books, 2 pens and 3 notebooks while Rashmi has 1 pen, 4 books and 5 notebooks in their respective school bags. Express the information as a matrix. What is the order of the matrix obtained?

Solution: The information can be represented in tabular form as

	Books	Pens	Notebooks
Shalabh	3	2	3
Rashmi	4	1	5

We can represent the given information as A = $\begin{bmatrix} 3 & 2 & 3 \\ 4 & 1 & 5 \end{bmatrix}$

Here first column shows the number of books, second column represents the number of pens and third column shows the number of notebooks the students have in their respective bags.

The order of matrix A is 2×3, having 6 elements.

Is that the only way to express the given information?

No, another orientation of the same information can be written as matrix B = $\begin{bmatrix} 3 & 4 \\ 2 & 1 \\ 3 & 5 \end{bmatrix}$

where first row denotes the number of books, second row denotes the number of pens while the third row denotes the number of notebooks each student has

The number of elements in matrix B is still 6 while the order is 3×2

Can you think of any other way to represent the information?

Example 2

Write the coordinates of triangle ABC with vertices A (4, -1), B (3, 2) and C (2, -4) in a matrix formation

Solution: Vertices of triangle ABC can be written in matrix form in two ways,

$$X = \begin{array}{c} A \\ B \\ C \end{array}\begin{bmatrix} 4 & -1 \\ 3 & 2 \\ 2 & -4 \end{bmatrix} \text{ of order } 3\times2 \text{ or } Y = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & -4 \end{bmatrix} \end{array} \text{ of order } 2\times3$$

Example 3

If a matrix has 4 elements, what are the possible orders such a matrix can have?

Solution: In a matrix is of order m × n, we have mn number of elements.

therefore, to find all possible orders of a matrix with 4 elements, we will find all ordered pairs of natural numbers, whose product is 4.

Hence, this matrix can have possible orders: 1 × 4, 4 ×1, 2 × 2

Example 4

Construct a 3 × 3 matrix whose elements are given by $a_{ij} = \frac{(i+2j)}{5}$

Solution: In general a 3 × 3 matrix is given by A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

For $a_{11}, i = 1 \; and \; j = 1 \Rightarrow a_{11} = \dfrac{(1+2.1)}{5} = \dfrac{3}{5}$

Similarly, For $a_{12}, i = 1 \; and \; j = 2 \Rightarrow a_{12} = \dfrac{(1+2.2)}{5} = 1$

Similarly, $a_{13} = \dfrac{(1+2.3)}{5} = \dfrac{7}{5}$

$$a_{21} = \frac{(2+2.1)}{5} = \frac{4}{5}$$

$$a_{22} = \frac{(2+2.2)}{5} = \frac{6}{5}$$

$$a_{23} = \frac{(2+2.3)}{5} = \frac{8}{5}$$

$$a_{31} = \frac{(3+2.1)}{5} = 1$$

$$a_{32} = \frac{(3+2.2)}{5} = \frac{7}{5}$$

$$a_{33} = \frac{(3+2.3)}{5} = \frac{9}{5}$$

Therefore $\quad A = \begin{bmatrix} \frac{3}{5} & 1 & \frac{7}{5} \\ \frac{4}{5} & \frac{6}{5} & \frac{8}{5} \\ 1 & \frac{7}{5} & \frac{9}{5} \end{bmatrix}$

As observed in all matrices discussed till now, we have seen that there can be different orientations of a matrix. Let us now discuss the different types of matrices.

2.3.1 TYPES OF MATRICES

1. **Rectangular matrix:** A matrix in which number of rows is not equal to number of columns is called a *Rectangular matrix.*

   For example, matrix A = $\begin{bmatrix} 2 & 5 \\ -1 & 2 \\ 4 & 0 \end{bmatrix}$ is a rectangular matrix of order 3×2

2. **Square Matrix**: A matrix in which the number of rows and the number of columns are equal is called a *Square matrix.*

   For example matrices B = $\begin{bmatrix} 0 & -4 & 3 \\ 1 & 0 & -7 \\ 2 & 2 & 0 \end{bmatrix}$ is a square matrix of order 3×3 or B is a square matrix of order 3.

3. **Row Matrix**: A matrix having exactly one row for a given number of columns is called a *Row matrix*. For example, Matrix D = $\begin{bmatrix} 2 & 3 \end{bmatrix}$ is a row matrix of order 1× 2

4. **Column Matrix**: A matrix having exactly one column for a given number of rows is called a *Column matrix*

   For example, Matrix C = $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ is a column matrix of order 3× 1

5. **Diagonal Matrix**: A square matrix in which all the non-diagonal entries are zero i.e., $a_{ij} = 0, \forall i \neq j$ is called a *Diagonal matrix*

   For example, P = $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ and Q = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ are diagonal matrices of order 2×2 and 3×3 respectively

6. **Scalar Matrix**: A diagonal matrix having same diagonal elements, i.e., $a_{ij} = k, \forall i = j$ where $k \neq 0$ is called a *Scalar matrix.*

For example, matrices A = $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ are scalar matrices of order 2×2 and 3 × 3 respectively

7. **Identity Matrix**: A scalar matrix in which all the diagonal entries are equal to 1, i.e. $a_{ij} = 1 \, \forall \, i = j \text{ and } a_{ij} = 0, \forall \, i \neq j$ is called an *Identity matrix, denoted by English alphabet I*

It is also known as *Unit matrix. A unit matrix of order n is written as* I_n.

Identity matrices $I = [1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are identity matrices of order 1×1, 2×2 and 3×3 respectively

Note that a scalar matrix with 1 as each diagonal element is an identity matrix

8. **Zero Matrix**: A matrix with each of its elements as zero, i.e. $a_{ij} = 0, \forall \leq i \leq m, 1 \leq j \leq n, \forall i, j \epsilon N$ is called as *zero matrix*

For example, matrix $[0], [0 \quad 0 \quad 0], \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are zero matrices of different orders. A zero matrix is denoted by English alphabet O, sometimes also called a null or void matrix as well.

9. **Equal Matrices:** Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ having same order m×n are called *Equal matrices* when each element of A is equal to the corresponding element of B,

i.e. $a_{ij} = b_{ij} \, \forall \, 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

In such a case we denote equal matrices as A = B

Example 5

If A = $\begin{bmatrix} 1 & 2a \\ -8 & b+1 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -6 \\ -8 & 13 \end{bmatrix}$ are equal matrices, find the values of a and b

Solution:

As A = B ⇒ $a_{ij} = b_{ij} \, \forall \, 1 \leq i \leq m \text{ and } 1 \leq j \leq n$ (∵ A = B)

Therefore, 2a = -6 ⇒ a = -3 and b + 1 = 13 ⇒ b = 12

Example 6

Find the values of a, b, c, and d from the following equation:

$$\begin{bmatrix} 4 & 24 \\ -3 & 11 \end{bmatrix} = \begin{bmatrix} 2a+b & 4c+3d \\ a-2b & 5c-d \end{bmatrix}$$

Solution: As the given matrices are equal,

therefore by equating the corresponding elements of both matrices we get:

$2a + b = 4$

$a - 2b = -3$

$4c + 3d = 24$

And $5c - d = 11$

Solving these equations, we get a = 1, b = 2, c = 3 and d = 4

2.3.2 CHECK YOUR PROGRESS

EXERCISE - A

1. Identify the type of matrices given below and write the order of each matrix:

i) $A = \begin{bmatrix} 2 & 3 \end{bmatrix}$

ii) $B = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

iii) $C = \begin{bmatrix} 2 & 5 \\ -1 & 2 \\ 4 & 0 \end{bmatrix}$

iv) $D = \begin{bmatrix} 1 & 3 & -4 \end{bmatrix}$

v) $E = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$

vi) $P = \begin{bmatrix} 0 & -4 & 3 \\ 1 & 0 & -7 \\ 2 & 2 & 0 \end{bmatrix}$

vii) $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

viii) $R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

ix) $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

x) $Z = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

2. i) $A = \begin{bmatrix} 0 & -4 & 3 \\ 1 & 0 & -7 \\ 2 & 2 & 0 \end{bmatrix}$, write the element a_{12}

ii) $B = \begin{bmatrix} -9 & 4 & -3 \\ -1 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix}$, find the sum of elements at b_{22} and b_{32}

iii) $C = \begin{bmatrix} -9 & 4 & -3 \\ -1 & 0 & 4 \\ 2 & 2 & 0 \end{bmatrix}$, find $c_{21}+c_{32}-c_{13}$

3. Construct matrix $A = [a_{ij}]$ of order 2X3 where $a_{ij} = \frac{(i+2j)^2}{2}$

4. Construct matrix $B = [b_{ij}]$ of order 2×2 where $b_{ij} = \frac{|i-j|}{3}$

5. How many distinct 2X2 matrices can be formed by using numbers 5, 7 and -1? Justify your answer.

6. A matrix has 14 elements. How many matrices of different orders are possible?

7. Find the values of a, b, c and d from the equation:

$$\begin{bmatrix} 14 & a+b \\ c+d & b+c \end{bmatrix} = \begin{bmatrix} a & -b \\ 8 & 0 \end{bmatrix}$$

2.4 ALGEBRA OF MATRICES

Now we are going to learn how to apply basic mathematical operations such as

- Multiplication of a matrix by a scalar value
- Addition and subtraction of two or more matrices

Multiplication of two or more matrices

2.4.1 MULTIPLICATION OF A MATRIX BY A SCALAR VALUE

A baker bakes two types of breads using the proportions of ingredients as follows:

This information can be denoted by:

$$A = \begin{array}{r} \\ Bread\ I \\ Bread\ II \end{array} \begin{array}{c} \begin{array}{ccc} Flour & Butter & Milk \end{array} \\ \begin{bmatrix} 2 & 50 & 20 \\ 3 & 25 & 10 \end{bmatrix} \end{array}$$

On a Sunday, the baker plans to make the breads thrice as much as he does on a regular day. We know that he will have to take thrice the quantity he uses for regular days.

Items	Flour	Butter	Milk
Bread I	2kg	50g	20ml
Bread II	3kg	25g	10ml

In this case, 3A = 3$\times$A

$$= 3 \times \begin{array}{r} \\ Bread\ I \\ Bread\ II \end{array} \begin{array}{c} \begin{array}{ccc} Flour & Butter & Milk \end{array} \\ \begin{bmatrix} 2 & 50 & 20 \\ 3 & 25 & 10 \end{bmatrix} \end{array}$$

Note that-

- *New matrix is obtained by multiplying each element of matrix A by 3*
- *Scalar multiplication is irrespective of the order of the given matrix*

$$= \begin{array}{r} \\ Bread\ I \\ Bread\ II \end{array} \begin{array}{c} \begin{array}{ccc} Flour & Butter & Milk \end{array} \\ \begin{bmatrix} 3\times 2 & 3\times 50 & 3\times 20 \\ 3\times 3 & 3\times 25 & 3\times 10 \end{bmatrix} \end{array}$$

$$= \begin{array}{r} \\ Bread\ I \\ Bread\ II \end{array} \begin{array}{c} \begin{array}{ccc} Flour & Butter & Milk \end{array} \\ \begin{bmatrix} 6 & 150 & 60 \\ 9 & 75 & 30 \end{bmatrix} \end{array}$$

Definition: For a matrix A = $[a_{ij}]$ of order m × n and k is a scalar quantity, then kA is another matrix obtained by multiplying each element of A by the scalar quantity k, i.e. kA = k $[a_{ij}]$ = $[ka_{ij}]$, $\forall\ 1 \le i \le m \text{ and } 1 \le j \le n$. The order of the new matrix remains the same as that of the order of the given matrix.

Properties of scalar Multiplication

If A = $[a_{ij}]$ and B = $[b_{ij}]$ are two matrices of the same order, say m × n, and k and p are scalars, then

(i) k (A +B) = k A + k B

(ii) (k + p) A = k A + p A

(iii) k (A – B) = kA – kB

Definition: For a non-zero matrix A, of order m×n, a matrix B of the same order is called *Negative matrix* of matrix A. If A + B = O, where O is the zero matrix of the same order. We denote negative matrix A as – A

For example, for A = $\begin{bmatrix} 11 & -b \\ -8c & 9 \end{bmatrix}$,

Negative A = - A = (-1) × $\begin{bmatrix} 11 & -b \\ -8c & 9 \end{bmatrix}$

$$= \begin{bmatrix} (-1)\times 11 & (-1)\times -b \\ (-1)\times -8c & (-1)\times 9 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & b \\ 8c & -9 \end{bmatrix}$$

Note that A + (-A) = $\begin{bmatrix} 11 & -b \\ -8c & 9 \end{bmatrix} + \begin{bmatrix} -11 & b \\ 8c & -9 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

2.4.2 ADDITION OF MATRICES

Let us consider a scenario where students of three sections of class XII are to be divided for two group activities in the math lab. The details of boys and girls for the activities are as given below:

Class	Number of boys		Number of girls	
	Group I	Group II	Group I	Group II
XII – A	6	4	8	6
XII – B	5	5	9	4
XII - C	8	4	7	5

Table 2.1(i)

Let matrix A represent group I students as G_1 = $\begin{array}{c c} & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 6 & 8 \\ 5 & 9 \\ 8 & 7 \end{bmatrix} \end{array}$

And matrix B, represent group II students as G_2 = $\begin{array}{c c} & \begin{array}{cc} \text{Boys} & \text{Girls} \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 4 & 6 \\ 5 & 4 \\ 4 & 5 \end{bmatrix} \end{array}$

Suppose the teacher in-charge needs a complied list of total number of boys and girls for both the activities, then we can represent it as an addition of matrices A and B as shown below:

$$G_1 + G_2 = \begin{array}{cc} & \begin{array}{cc} \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 6 & 8 \\ 5 & 9 \\ 8 & 7 \end{bmatrix} \\ & \begin{array}{cc} Boys & Girls \end{array} \end{array} + \begin{array}{cc} & \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 4 & 6 \\ 5 & 4 \\ 4 & 5 \end{bmatrix} \\ & \begin{array}{cc} Boys & Girls \end{array} \end{array}$$

$$= \begin{array}{cc} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 6+4 & 8+6 \\ 5+5 & 9+4 \\ 8+4 & 7+5 \end{bmatrix} \\ & \begin{array}{cc} Boys & Girls \end{array} \end{array} = \begin{array}{cc} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 10 & 14 \\ 10 & 13 \\ 12 & 12 \end{bmatrix} \\ & \begin{array}{cc} Boys & Girls \end{array} \end{array}$$

Note that-

- *Both the matrices are of equal order*
- *The sum of matrices is obtained by adding corresponding elements of the matrices*

Definition: For two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$, of same order m × n, the sum of two matrices A and B is defined as a matrix S = A+B = $[s_{ij}]$ of order m ×n such that

$$s_{ij} = a_{ij} + b_{ij} \ \forall\ 1 \le i \le m \text{ and } 1 \le j \le n$$

Properties of addition of matrices

i) Addition of two or more matrices is possible only when the given matrices are of the same order. The order of resultant matrix is also same on the order of the given matrices.

$\Rightarrow A_{m\times n} + B_{m\times n} = C_{m\times n}$

ii) Matrix addition is Commutative $\Rightarrow A_{m\times n} + B_{m\times n} = B_{m\times n} + A_{m\times n}$

iii) Matrix addition is Associative $\Rightarrow A_{m\times n} + (B_{m\times n} + C_{m\times n}) = (A_{m\times n} + B_{m\times n}) + C_{m\times n}$

iv) Zero matrix is the additive identity $\Rightarrow A_{m\times n} + O_{m\times n} = A_{m\times n} = O_{m\times n} + A_{m\times n}$

v) Negative of a matrix is the additive inverse of the given matrix.

$\Rightarrow A_{m\times n} + (-A_{m\times n}) = O_{m\times n}$

2.4.3 SUBTRACTION OF MATRICES

From the table 2.1(i), if we need to know how many more boys and girls are there in group I as compared to group II

Clearly that would need us to find $G_1 - G_2$ matrix

Definition: For two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$, of same order m × n, the difference of the two matrices A and B is defined as a matrix D = A - B = $[d_{ij}]$ of order m ×n such that $d_{ij} = a_{ij} - b_{ij}$ $\forall\ 1 \le i \le m \text{ and } 1 \le j \le n$

From table 2.1(i), $G_1 - G_2 = \begin{array}{cc} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 6 & 8 \\ 5 & 9 \\ 8 & 7 \end{bmatrix} \\ & \begin{array}{cc} Boys & Girls \end{array} \end{array} - \begin{array}{cc} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 4 & 6 \\ 5 & 4 \\ 4 & 5 \end{bmatrix} \\ & \begin{array}{cc} Boys & Girls \end{array} \end{array}$

$$= \begin{array}{cc} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 6-4 & 8-6 \\ 5-5 & 9-4 \\ 8-4 & 7-5 \end{bmatrix} \\ & \begin{array}{cc} Boys & Girls \end{array} \end{array} = \begin{array}{cc} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 2 & 2 \\ 0 & 5 \\ 4 & 2 \end{bmatrix} \\ & \begin{array}{cc} Boys & Girls \end{array} \end{array}$$

Example 7

For the matrices A = $\begin{bmatrix} 3 & 4 & 0 \\ -1 & 12 & 3 \\ 6 & 1 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 7 & 7 & 2 \\ -11 & 0 & 2 \\ 3 & -1 & 4 \end{bmatrix}$ and

C = $\begin{bmatrix} 11 & 2 & -3 \\ 2 & 0 & 1 \\ 1 & 9 & 1 \end{bmatrix}$, calculate

i) 2A

ii) A + B – C

iii) 3B – A + 2C

Solution:

i) $2A = 2 \times \begin{bmatrix} 3 & 4 & 0 \\ -1 & 12 & 3 \\ 6 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 0 \\ -2 & 24 & 6 \\ 12 & 2 & 4 \end{bmatrix}$

ii) $A + B - C = \begin{bmatrix} 3 & 4 & 0 \\ -1 & 12 & 3 \\ 6 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 7 & 7 & 2 \\ -11 & 0 & 2 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 11 & 2 & -3 \\ 2 & 0 & 1 \\ 1 & 9 & 1 \end{bmatrix} =$

$$\begin{bmatrix} 3+7-11 & 4+7-2 & 0+2-(-3) \\ -1+(-11)-2 & 12+0-0 & 3+2-1 \\ 6+3-1 & 1+(-1)-9 & 2+4-1 \end{bmatrix} = \begin{bmatrix} -1 & 9 & 5 \\ -14 & 12 & 4 \\ 8 & -9 & 5 \end{bmatrix}$$

iii) $3B\text{-}A\text{+}2C = 3 \times \begin{bmatrix} 7 & 7 & 2 \\ -11 & 0 & 2 \\ 3 & -1 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 4 & 0 \\ -1 & 12 & 3 \\ 6 & 1 & 2 \end{bmatrix} + 2 \times \begin{bmatrix} 11 & 2 & -3 \\ 2 & 0 & 1 \\ 1 & 9 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 21-3+22 & 21-4+4 & 6-0-6 \\ -33+1+4 & 0-12+0 & 6-3+2 \\ 9-6+2 & -3-1+18 & 12-2+2 \end{bmatrix}$$

$$= \begin{bmatrix} 40 & 21 & 0 \\ -28 & -12 & 5 \\ 5 & 14 & 12 \end{bmatrix}$$

Example 8

If X = $\begin{bmatrix} -1 & 3 \\ 8 & 4 \end{bmatrix}$ and Y = $\begin{bmatrix} -5 & 1 \\ -1 & -2 \end{bmatrix}$ then find the matrix Z, such that 2X + Y = 5Z

Solution: As 2X+Y = 5Z

Then order of the matrix on LHS and RHS must be same

Let matrix Z be of order 2 × 2 written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $2 \times \begin{bmatrix} -1 & 3 \\ 8 & 4 \end{bmatrix} + \begin{bmatrix} -5 & 1 \\ -1 & -2 \end{bmatrix} = 5 \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} -7 & 7 \\ 15 & 6 \end{bmatrix} = \begin{bmatrix} 5a & 5b \\ 5c & 5d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{-7}{5} & \frac{7}{5} \\ 3 & \frac{6}{5} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Example 9

Find X and Y if X+Y = $\begin{bmatrix} -1 & 13 \\ 2 & 4 \end{bmatrix}$ and X − Y = $\begin{bmatrix} 5 & -8 \\ 3 & 0 \end{bmatrix}$

Solution: (X+Y) +(X-Y) = $\begin{bmatrix} -1 & 13 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -8 \\ 3 & 0 \end{bmatrix}$

$\Rightarrow \quad 2X = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 2 & \frac{5}{2} \\ \frac{5}{2} & 2 \end{bmatrix}$

Also, (X+Y)-(X-Y) = $\begin{bmatrix} -1 & 13 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & -8 \\ 3 & 0 \end{bmatrix}$

$\Rightarrow \quad 2Y = \begin{bmatrix} -6 & 21 \\ -1 & 4 \end{bmatrix} \Rightarrow Y = \begin{bmatrix} -3 & \frac{21}{2} \\ \frac{-1}{2} & 2 \end{bmatrix}$

2.4.4 MULTIPLICATION OF MATRICES

Here is another application of matrix algebra in a real-life situation, Neha private limited company hired a public-relations firm to promote their business by distributing free sample of two products. The cost of promotion per sample of each product (in rupees) is given by:

$$C = \begin{array}{l} \text{Product I} \\ \text{Product II} \end{array} \begin{bmatrix} 4.50 \\ 5 \end{bmatrix}$$
$$\text{Cost per sample}$$

The number of products of each type promoted among men and women in a city is given by matrix X =

$$\begin{array}{l} \text{Men} \\ \text{Women} \end{array} \begin{bmatrix} 20000 & 10000 \\ 15000 & 21000 \end{bmatrix}$$
$$\text{Product I} \quad \text{Product II}$$

Using matrix algebra, find the total amount spent by the company for their promotion campaign

The total cost of promoting among men = $20000 \times 4.50 + 10000 \times 5$

= 90000 + 50000 = 140000

And, total cost of promoting among women = $15000 \times 4.50 + 21000 \times 5$

= 67500 + 105000 = 172500

In this case of multiplication of two matrices X and C, the number of columns in X should be equal to the number of rows in C.

Furthermore, for getting the elements of the product matrix, we take rows of A and columns of B, multiply them element-wise and take the sum as shown below;

Using matrix algebra, we can find XC =

$$\begin{array}{l} \text{Men} \\ \text{Women} \end{array} \begin{bmatrix} 20000 & 10000 \\ 15000 & 21000 \end{bmatrix} \times \begin{array}{l} \text{Product I} \\ \text{Product II} \end{array} \begin{bmatrix} 4.50 \\ 5 \end{bmatrix}$$
$$\text{Product I} \quad \text{Product II} \qquad\qquad \text{Cost per contact}$$

$$= \begin{array}{l} Promotion\ among\ men \\ Promotion\ among\ womenn \end{array} \begin{bmatrix} 20000 \times 4.50 + 10000 \times 5 \\ 15000 \times 4.50 + 21000 \times 5 \end{bmatrix}$$
$$\qquad\qquad\qquad\qquad\qquad Cost\ (in\ rupees)$$

$$= \begin{bmatrix} 140000 \\ 172500 \end{bmatrix}$$

Definition: For two matrices A = $[a_{ij}]$ of order m×n and B = $[b_{jk}]$ of order n × p, the multiplication of the matrices AB is defined as a matrix P = AB = $[p_{ik}]$ of order m ×p such that for finding the element p_{ik},we multiple ith row of first matrix A with the kth column of second matrix B and calculate the sum of these products

i.e. $p_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + a_{i4}b_{4k} + \cdots + a_{in}b_{nk}\ \forall\ 1 \le i \le m\ and\ 1 \le j \le n$

Example 10

Let A = $\begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix}$, find AB and BA.

Solution: Before finding the product of two matrices, it is important to check if the order of multiplication is well defined.

Since the number of columns in matrix A is equal to the number of rows in matrix B, also the number of columns in matrix B is equal to the number of rows in matrix A, products AB and BA are defined and each will be of order 2×2.

Hence AB = $\begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix} \begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix}$

$$= \begin{bmatrix} (3)(-9) + (5)(1) & (3)(2) + (5)(-7) \\ (-4)(-9) + (6)(1) & (-4)(2) + (6)(-7) \end{bmatrix} = \begin{bmatrix} -22 & -29 \\ 42 & -50 \end{bmatrix}$$

BA = $\begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$

$$= \begin{bmatrix} (-9)(3) + (2)(-4) & (-9)(5) + (2)(6) \\ (1)(3) + (-7)(-4) & (1)(5) + (-7)(6) \end{bmatrix}$$

$$= \begin{bmatrix} -35 & -33 \\ 31 & -37 \end{bmatrix}$$

If A, B are matrices of orders respectively m × n, k × l , then both AB and BA are defined if and only if n = k and l = m.

Observe that in this case AB ≠ BA

This observation does not mean that defined multiplication of matrices is not commutative

For example, let A = $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$

Clearly, AB and BA are defined and AB = $\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$ = BA

- Multiplication of diagonal matrices is commutative

Recall that for two real numbers 'x' and 'y' when xy = o then either x = 0 or y = 0.

Let us see if is true in the case of multiplication of matrices as well

Consider two non-zero matrices A = $\begin{bmatrix} 0 & -9 \\ 0 & 4 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix}$

Here AB = $\begin{bmatrix} 0 & -9 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Whereas neither of the two matrices A or B are zero matrices. Thus, if the product of two matrices is a zero matrix, then it is not necessary that one of the matrices is a zero matrix

Example 11

Find $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 3 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 \\ 9 & 3 & -3 \\ 6 & 1 & -2 \end{bmatrix} = \begin{bmatrix} -2-9+12 & 0-3+2 & 1+3-4 \\ 0+18-18 & 0+6-3 & 0-6+6 \\ -6-18+24 & 0-6+4 & 3+6-8 \end{bmatrix}$

$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

Properties of multiplication of matrices

i) The multiplication of matrices is associative i.e. for any three matrices A, B and C, (AB) C = A (BC), whenever order of multiplication is defined on both sides.

ii) Distributive property of multiplication holds true for multiplication of matrices. i.e. For three matrices A, B and C

A (B+C) = AB + AC

and, A (B - C) = AB - AC, whenever order of multiplication is defined on both sides.

iii) For a given square matrix A of order m×m, there exists a multiplicative identity matrix I, of the same order such that IA = AI = A.

Example 12

For A = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, prove that $A^2 - 4A + I = O$, where O is a zero matrix·

Solution: For matrix A of order 2× 2, we will take I identity matrix of the same order

Now $A^2 = A.A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 7-8+1 & 12-12+0 \\ 4-4+0 & 7-8+1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Example 13

Given that $X_{2\times n}$, $Y_{3\times k}$, $Z_{2\times p}$, $W_{n\times 3}$ and $P_{p\times k}$ are matrices of specified order. What are the conditions

i. for n, k and p so that 3PY + 2WY is defined

ii. for the order of the matrix 2X – 3Z

Solution:

i. For defined multiplication of matrices P and Y, number of columns of matrix P must be equal to number of rows of matrix Y $\Rightarrow k = 3$

Then Order of PY = $p \times k$

Also order of WY = $n\times k$

Then for 3PY + 2WY to be defined the order of PY and WY must be same

$\Rightarrow p \times k = n\times k \Rightarrow p = n$

ii. 2X – 3Z is defined if order X is equal to order of Z $\Rightarrow$ $2\times n = 2\times p \Rightarrow p = n$

2.5 SPECIAL MATRICES

2.5.1 TRANSPOSE OF A MATRIX

Definition: For a matrix $A = [a_{ij}]$ of order m×n, the matrix obtained by interchanging the rows and columns of the matrix A is called the transpose of matrix A.

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ then $A' = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix}$

$B = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 3 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ then $B' = \begin{bmatrix} -2 & 9 & 6 \\ 0 & 3 & 1 \\ 1 & -3 & -2 \end{bmatrix}$

Transpose of a matrix A is denoted by A′ or A^T.

$P = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ then $P' = \begin{bmatrix} 1 & 3 \\ -2 & 2 \\ 3 & -1 \end{bmatrix}$

Example 14

For a matrix A of order mn, the order of transpose matrix A′ is n m.

If $A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix}$ then find the matrix A' -2B

Solution: $A' - 2B = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} - 2\begin{bmatrix} -2 & 3 \\ 5 & -4 \end{bmatrix}$

$$= \begin{bmatrix} 2+4 & 1-6 \\ 3-10 & -4+8 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix}$$

Properties of Transpose of a matrix

For given matrices A and B:

i. $(A')' = A$

ii. $(kA)' = kA'$ (where k is any constant)

iii. $(A + B)' = A' + B'$

iv. $(A\ B)' = B'\ A'$

2.5.2 SYMMETRIC AND SKEW SYMMETRIC MATRICES

Definition: For a given square matrix $A = [a_{ij}]$, if $A' = A$, $\forall\ 1 \le i \le m \text{ and } 1 \le j \le n$, then the matrix A is called a **symmetric matrix.**

For example, $P = \begin{bmatrix} \sqrt{3} & -8 \\ -8 & 9 \end{bmatrix}$ is a symmetric matrix as P′ = P?

$A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & -1 & -2 \\ 3 & -2 & 1 \end{bmatrix}$ is a symmetric matrix as A′ = A?

Definition: For a given square matrix $A = [a_{ij}]$,

if $A' = -A$, $\forall\ 1 \le i \le m \text{ and } 1 \le j \le n$, then the matrix A is called a **skew-symmetric matrix**.

All diagonal elements of a skew-symmetric matrix are zero

For example, $B = \begin{bmatrix} 0 & -8 \\ 8 & 0 \end{bmatrix}$ and $B' = \begin{bmatrix} 0 & 8 \\ -8 & 0 \end{bmatrix} = -B$

⇒ is a skew-symmetric matrix.

For $Q = \begin{bmatrix} 0 & -2 & -8 \\ 2 & 0 & 4 \\ 8 & -4 & 0 \end{bmatrix}$, $Q' = \begin{bmatrix} 0 & 2 & 8 \\ -2 & 0 & -4 \\ -8 & 4 & 0 \end{bmatrix}$ = -Q

⇒ Q is a skew-symmetric matrix

For a square matrix A having real values as elements,

- $A + A'$ is a symmetric matrix
- $A - A'$ is a skew symmetric matrix.

Example 15

If $P = \begin{bmatrix} cosx & sinx \\ -sinx & cosx \end{bmatrix}$, then verify that P′P = I, where I is an identity matrix

Solution: $P = \begin{bmatrix} cosx & sinx \\ -sinx & cosx \end{bmatrix} \Rightarrow P' = \begin{bmatrix} cosx & -sinx \\ sinx & cosx \end{bmatrix}$

$$\Rightarrow P'P = \begin{bmatrix} cosx & sinx \\ -sinx & cosx \end{bmatrix}\begin{bmatrix} cosx & -sinx \\ sinx & cosx \end{bmatrix}$$

$$= \begin{bmatrix} cos^2x + sin^2x & -cosxsinx + cosxsinx \\ -cosxsinx + cosxsinx & cos^2x + sin^2x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 16

If A, B are symmetric matrices of same order, then what can be said for matrix AB – BA ?

Solution: As A, B are symmetric matrices

$\Rightarrow A' = A$ and $B' = B$ ————(i)

Now $(AB - BA)' = (AB)' - (BA)'$

$= (B'A') - (A'B')$

$= BA - AB$ [from (i)]

$= -(AB - BA)$

$\Rightarrow$ AB – BA is a skew-symmetric matrix

Example 17

Express the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

Solution: $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix}$

$$A' = \begin{bmatrix} 1 & -4 & 3 \\ 2 & -1 & 5 \\ 3 & 0 & 1 \end{bmatrix}$$

Let $P = \frac{1}{2}(A + A') = \frac{1}{2}\left[\begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -4 & 3 \\ 2 & -1 & 5 \\ 3 & 0 & 1 \end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix} 2 & -2 & 6 \\ -2 & -2 & 5 \\ 6 & 5 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1 & 3 \\ -1 & -1 & \frac{5}{2} \\ 3 & \frac{5}{2} & 1 \end{bmatrix}$$

Also $P' = \begin{bmatrix} 1 & -1 & 3 \\ -1 & -1 & \frac{5}{2} \\ 3 & \frac{5}{2} & 1 \end{bmatrix} = P$

$\Rightarrow$ P is a symmetric matrix

Let $Q = \frac{1}{2}(A - A') = \frac{1}{2}\left[\begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -4 & 3 \\ 2 & -1 & 5 \\ 3 & 0 & 1 \end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix} 0 & 6 & 0 \\ -6 & 0 & -5 \\ 0 & 5 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & \frac{-5}{2} \\ 0 & \frac{5}{2} & 0 \end{bmatrix}$$

$$\text{Also } Q' = = \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & \frac{5}{2} \\ 0 & \frac{-5}{2} & 0 \end{bmatrix} = -Q$$

$$\Rightarrow Q \text{ is a skew} - \text{symmetric matrix}.$$

$$\text{Also } P + Q = \begin{bmatrix} 1 & -1 & 3 \\ -1 & -1 & \frac{5}{2} \\ 3 & \frac{5}{2} & 1 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 0 \\ -3 & 0 & \frac{-5}{2} \\ 0 & \frac{5}{2} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -4 & -1 & 0 \\ 3 & 5 & 1 \end{bmatrix} = A$$

A is represented as the sum of a symmetric and a skew- symmetric matrix.

2.5.3 CHECK YOUR PROGRESS

EXERCISE–B

1. Complete the following table.

Order of the matrix

A	B	A$\pm$B	AB
2 × 2	2 × 2		
2 × 3	3 × 2		
3 × 4	4 × 1		
3 × 3	3 × 3		
2 × 3		2 × 3	
	3 × 2		1 × 2
2 × 3		2 × 3	
1 × 3	3 × 2		

2. For A=$\begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix}$, B =$\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$and C=$\begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}$, show that

   i. Commutative property does not hold true for multiplication of matrices A and B i.e. AB$\neq$BA

   ii. Associative property holds true for multiplication of three matrices, i.e. A(BC)=(AB)C

3. Consider A= $\begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 2 \\ 3 & -2 & 1 \end{bmatrix}$ verify the A.I = I.A=A, where I is the identity matrix of order 3×3

4. If A=$\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ and B =$\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$then show that

   i. $(A+B)'=(A)'+(B)'$

   ii. $(AB)' = (B)'(A)'$

5. Do as directed

i) For A = $\begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix}$, find A^2 - 6A

ii) Evaluate $\begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

iii) Find a matrix A such that $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - A$

iv) If the matrix X = $\begin{bmatrix} -1 & 5 \\ 0 & 13 \end{bmatrix}$ is equal to the matrix Y = $\begin{bmatrix} p-q & 2p+r \\ 2p-q & 3r+s \end{bmatrix}$ then find value of p, q, r and s.

v) Let P = $\begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$ and Q=$\begin{bmatrix} 1 & 4 \\ 7 & 2 \end{bmatrix}$ then calculate 3P-2Q.

6. If A = $\begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$, then find a matrix C, such that 3A – 2B + 4C = 0

7. Given A=$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, B=$\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, find:

i. 2A – 3B
ii. AB
iii. BA
iv. AB - BA

8. For A = $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ show that $A^3 - 23A - 40\,I = O$, where I is an identity matrix of order 3, and O is zero matrix

9. Two booksellers A and B sell the textbook of Mathematics and Applied Mathematics. In the month of march, bookseller A sold 250 books of Mathematics and 400 books of Applied Mathematics whereas bookseller B sold 230 books of Mathematics and 425 books of Applied Mathematics. In the month of April, bookseller A sold 550 books of Mathematics and 300 books of Applied Mathematics and bookseller B sold 270 books of Mathematics and 450 books of Applied Mathematics.

Represent the given information into matrix form and find the total sale for both the booksellers in the month of March and April, using matrix algebra.

10. Cost of a pen and a notebook are Rs.12 and Rs. 27 respectively. On a given day, shopkeeper P sells 5 pens and 7 notebooks, whereas shopkeeper Q sells 6 pens and 4 notebooks on a particular day. Find the income of both the shopkeepers, using matrix algebra

2.6 DETERMINANT

Recall that for a given system of linear equations

i.e. $a_1 x + b_1 y = c_1$

and, $a_2 x + b_2 y = c_2$

the system of equations has a unique solution if $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \Rightarrow a_1 b_2 \neq a_2 b_1$

$\Rightarrow a_1 b_2 - a_2 b_1 = 0$

In this chapter we have learnt that. $a_1 x + b_1 y = c_1$

and, $a_2 x + b_2 y = c_2$ can be represented in matrix form as:

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The expression $a_1 b_2 - a_2 b_1$ is associated with the square matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called determinant of the matrix and denoted as *det*A or |A| and **sometimes by Greek symbol** Δ.

Every square matrix is associated with a number C real or complex, known as Determinant of the matrix A and is denoted by detA or |A|, or A determinant is also expressed using greek symbol Δ.

Definition: For a given square matrix A = $[a_{ij}]$, of order m, a number (real or complex) is known as determinant of the matrix A

For matrix A = [a] of order 1 × 1
*det*A = a

Let X be a set of square matrices and R be the set of numbers (real or complex) such that a function f is defined as f : M → K by f (A) = k, where A ∈ X and k ∈ R, then f(A) is called the determinant of A.

Only a square matrix has a determinant value associated to it

For example, For A = [2], a matrix of order 1×1

det A = |A| = 2

2.6.1 MINOR OF AN ELEMENT

Definition: In a given determinant, minor of an element a_{ij} is the determinant obtained by deleting its *i*th row and *j*th column in which element a_{ij} lies, and denoted by M_{ij}.

Example 18:

Find the minor of element -11 in A = $\begin{vmatrix} 6 & -11 \\ -1 & 3 \end{vmatrix}$

Minor of an element of a determinant of order m(m ? 2) is a determinant of order n − 1

Solution: As -11 is $(1, 2)^{th}$ element of the matrix

We will eliminate first row and second column $\begin{vmatrix} 6 & -11 \\ -1 & 3 \end{vmatrix}$ and M_{12} = -1

For a matrix A = $\begin{bmatrix} 6 & -5 \\ -7 & 4 \end{bmatrix}$ of order 2×2,

determinant of A = *det* A = |A| is represented as $\begin{vmatrix} 6 & -5 \\ -7 & 4 \end{vmatrix}$

And calculated as |A| = $\begin{vmatrix} 6 & -5 \\ -7 & 4 \end{vmatrix}$

= [6×4] – [(–5)×(–7)]

= 24 – 35

= –11.

Example 19

Given A = $\begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix}$, find the following

i) |A| ii) |B| iii) 2|A| iv) |2A| v) |A| |B| vi) |AB|

Solution:

i) $|A| = \begin{vmatrix} 3 & 5 \\ -4 & 6 \end{vmatrix}$ = (3)(6)-(-4)(5)= 18+20=38

ii) $|B| = \begin{vmatrix} -9 & 2 \\ 1 & -7 \end{vmatrix}$ =(-9)(-7)-(1)(2)=63-2=61

iii) $2|A| = 2\begin{vmatrix} 3 & 5 \\ -4 & 6 \end{vmatrix}$ = 2 (38)= 76

iv) $A\begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$, $2A = 2\begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix} = \begin{bmatrix} 6 & 10 \\ -8 & 12 \end{bmatrix}$

$|2A| = \begin{vmatrix} 6 & 10 \\ -8 & 12 \end{vmatrix}$ = (6) (12)-(-8) (10) =72+80=152

v) $|A|\ |B| = \begin{vmatrix} 3 & 5 \\ -4 & 6 \end{vmatrix} \begin{vmatrix} -9 & 2 \\ 1 & -7 \end{vmatrix}$ =38. 61 =2318

vi) $AB = \begin{bmatrix} -22 & -29 \\ 42 & -50 \end{bmatrix}$,

$|AB| = \begin{vmatrix} -22 & -29 \\ 42 & -50 \end{vmatrix}$ = (-22) (-50) - (42) (-29) = 1100+1218 = 2318

Example 20

Find x if $\begin{vmatrix} 3 & -6 \\ 4 & 0 \end{vmatrix} = \begin{vmatrix} 3 & x^2 \\ x & -1 \end{vmatrix}$

Solution: As $\begin{vmatrix} 3 & -6 \\ 4 & 0 \end{vmatrix} = \begin{vmatrix} 3 & x^2 \\ x & -1 \end{vmatrix}$

$\Rightarrow$ 0 + 24 = -3 - x^3

$\Rightarrow x^3$ = -27

$\therefore x = -3$

Example 21

Find the minor of element 7 in the determinant B = $\begin{vmatrix} 2 & 3 & 0 \\ -3 & 1 & 7 \\ 1 & -2 & 5 \end{vmatrix}$

Solution: As 7 is $(2,3)^{th}$ element of the determinant $\begin{vmatrix} 2 & 3 & 0 \\ -3 & 1 & 7 \\ 1 & -2 & 5 \end{vmatrix}$

We shall eliminate second row and the third column and M_{23} = $\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$ = 2×(-2) - 3×1 = - 4 – 3 = -7

2.6.2 COFACTOR OF AN ELEMENT OF A DETERMINANT

Definition: In a square matrix, cofactor of an element a_{ij} , denoted by A_{ij} or C_{ij} is defined by $A_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij}

In example 17, $A_{12} = (-1)^{1+2}M_{12}$ = (-1) (-4) = 1

In example 21, $A_{23} = (-1)^{2+3}M_{23}$ = $(-1)^5(-7)$ = 7

To evaluate a determinant, elements of a row or a column are multiplied with their corresponding cofactors.

For a determinant of order 3X3, the value of the determinant can be evaluated by expanding along the first row as $\mathbf{a_{11}\ A_{11} + a_{12}\ A_{12} + a_{13}\ A_{13}}$..

Now let us consider another square matrix B = $[b_{ij}]$, of order 3×3

let B = $\begin{bmatrix} 2 & -1 & -3 \\ -3 & 0 & 5 \\ 1 & -1 & -2 \end{bmatrix}$

For a matrix B of order 3×3, *det* B is calculated in terms of determinants of order 2×2. We can expand and calculate *det* B in six ways by expanding with respect to any one of the three rows (R_1 , R_2 and R_3) or any one of the three columns (C_1 , C_2 and C_3) giving the same determinant value

Here, we shall multiply the elements of any one row or column by their respective cofactors. For example, if we use R_1 to find *det* B, it will be as follows:

$$det\ B = \begin{vmatrix} 2 & -1 & -3 \\ -3 & 0 & 5 \\ 1 & -1 & -2 \end{vmatrix} = a_{11}\ A_{11} + a_{12}\ A_{12} + a_{13}\ A_{13}$$

$$= 2\begin{vmatrix} 0 & 5 \\ -1 & -2 \end{vmatrix} + 1\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} - 3\begin{vmatrix} -3 & 0 \\ 1 & -1 \end{vmatrix}$$

$$= 2[\ 0 - (-5)\ ] + 1[\ 6 - 5\ ] - 3[\ 3 - 0\ ]$$

$$= 10 + 1 - 9 = 0$$

If the determinant of any matrix is zero then that matrix is called Singular Matrix.

Example 22

Evaluate $\Delta = \begin{vmatrix} 0 & cosx & -sinx \\ -cosx & 0 & cosy \\ sinx & -cosy & 0 \end{vmatrix}$

Solution: Expanding along R_1

Δ = 0 – cos x (0 – sin x cos y) – sin x (cos x cos y – 0) = sin x cos x cos y – sinx cosx cos y = 0

Example 23

When elements of a row or a column of a determinant are multiplied with cofactors of any other row or column, then their sum is always zero

Find minors and cofactors of the elements of the determinant $A = \begin{vmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{vmatrix}$ and check whether $a_{11}\ A_{31} + a_{12}\ A_{32} + a_{13}\ A_{33} = 0$

Solution: $M_{11} = \begin{vmatrix} 0 & 4 \\ 5 & -7 \end{vmatrix} = 0 - 20 = -20$ and $A_{11} = (-1)^{1+1}M_{11} = -20$

$M_{12} = \begin{vmatrix} 6 & 4 \\ 1 & -7 \end{vmatrix} = -42-4 = -46$ and $A_{12} = (-1)^{1+2}M_{12} = 46$

$M_{13} = \begin{vmatrix} 6 & 0 \\ 1 & 5 \end{vmatrix} = 30 - 0 = 30$ and $A_{13} = (-1)^{1+3}M_{13} = 30$

$M_{21} = \begin{vmatrix} -3 & 5 \\ 5 & -7 \end{vmatrix} = 21 - 25 = -4$ and $A_{21} = (-1)^{2+1}M_{21} = 4$

$M_{22} = \begin{vmatrix} 2 & 5 \\ 1 & -7 \end{vmatrix} = -14 - 5 = -19$ and $A_{22} = (-1)^{2+2}M_{22} = -19$

$M_{23} = \begin{vmatrix} 2 & -3 \\ 1 & 5 \end{vmatrix} = 10 + 3 = 13$ and $A_{23} = (-1)^{2+3}M_{23} = -13$

$M_{31} = \begin{vmatrix} -3 & 5 \\ 0 & 4 \end{vmatrix} = -12 - 0 = -12$ and $A_{31} = (-1)^{3+1}M_{31} = -12$

$M_{32} = \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} = 8 - 30 = -22$ and $A_{32} = (-1)^{3+2}M_{32} = 22$

$M_{33} = \begin{vmatrix} 2 & -3 \\ 6 & 0 \end{vmatrix} = 0 + 18 = 18$ and $A_{33} = (-1)^{3+3}M_{33} = 18$

Now $a_{11}\ A_{31} + a_{12}\ A_{32} + a_{13}\ A_{33} = 2\ (-12) - 3\times22 + 5\times18 = 0$

2.6.3 ADJOINT OF A MATRIX

Definition: The transpose of cofactors of a matrix of a square matrix A = $[a_{ij}]$ is called Adjoint matrix of matrix A and is denoted by *adj* A.

Using the cofactor values of corresponding elements of given matrix A in example above, we can write *adj* A = $\begin{bmatrix} -20 & 46 & 30 \\ 4 & -19 & -13 \\ -12 & 22 & -18 \end{bmatrix}' = \begin{bmatrix} -20 & 4 & -12 \\ 46 & -19 & 22 \\ 30 & -13 & -18 \end{bmatrix}$

Properties of adjoint of a matrix

For a given square matrix A, of order n,

i. $A(\text{adj } A) = (\text{adj } A)\, A = |A|\, I$, where I is the identity matrix of order n

ii. $|adj(A)| = |A|^{n-1}$

Example 24

Find k, if $A=\begin{bmatrix} -2 & 3 \\ k & 4 \end{bmatrix}$ is a singular matrix

Solution: As A is a singular matrix $\Rightarrow |A| = 0$

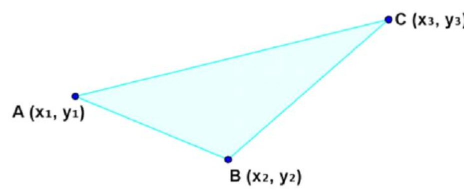
$$\Rightarrow -8 - 3k = 0$$

$$\Rightarrow \quad k = \frac{-8}{3}$$

2.6.4 AREA OF TRIANGLE

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC, then

Area $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$



i. Since area is a positive quantity, we always take the absolute value of the determinant

ii. If area is given, use both positive and negative values of the determinant for calculation.

iii. For given vertices with coordinates $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$; if the area of a triangle is zero the points A, B and C are collinear

Example 25

Find the area of the triangle with vertices A (5, 4), B (2, -6) and C (-2, 4).

Solution: Area $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

$$= \frac{1}{2}\begin{vmatrix} 5 & 4 & 1 \\ 2 & -6 & 1 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}(-70) = -35$$

Hence area of the triangle = $|-35|$ = 35 sq. units

Example 26

For what value of k, points P (3,-2), Q (8, 8) and R (k, 2) are collinear.

Solution: As the points P, Q and R are collinear

$\Rightarrow$ Area of triangle PQR = $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 3 & -2 & 1 \\ 8 & 8 & 1 \\ k & 2 & 1 \end{vmatrix} = 0$$

Solving above determinant, we get, | 3(8-2) +2(8-k) +1(16-8k)| = 0

|18 + 16 – 2k +16 -8k | = 0

$\Rightarrow$ |-10k + 50 | = 0

$\Rightarrow$ $\pm$(-10k + 50) = 0 $\Rightarrow$ k = 5

2.6.5 PROPERTIES OF A DETERMINANT

(i) For any two square matrices A and B, multiplication of their determinant values is same as determinant of their product

For example, let us consider two matrices A = $\begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$ and B = $\begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix}$

$|A|\ |B| = \begin{vmatrix} 3 & 5 \\ -4 & 6 \end{vmatrix}\begin{vmatrix} -9 & 2 \\ 1 & -7 \end{vmatrix} = 38\times 61 = 2318$

And, AB = $\begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} -9 & 2 \\ 1 & -7 \end{bmatrix} = \begin{bmatrix} -22 & -29 \\ 42 & -50 \end{bmatrix}$

$\Rightarrow |AB| = \begin{vmatrix} -22 & -29 \\ 42 & -50 \end{vmatrix}$ = (-22) ×(-50) - (42)×(-29) = 1100+1218 = 2318

Therefore, we can say that for square matrices A and B, |A|×|B| = |AB|

(ii) If two square matrices A and B of order n are such that A = kB, then | A| = k^n | B |, where n = 1, 2, 3,...

Let us consider two matrices A = $\begin{bmatrix} -4 & 12 \\ 20 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} -1 & 3 \\ 5 & 0 \end{bmatrix}$ of order 2×2

Clearly A = 4B

And, |A| = 0 – 240 = -240

|B| = 0 – 15 = -15

$\Rightarrow$ |A| = 16|B| = 4^2 |B|, where k = 4 and n = 2

iii) The value of the determinant remains unchanged if its rows and columns are interchanged

Consider $\Delta = \begin{vmatrix} 5 & -3 & 1 \\ 0 & 2 & -3 \\ 2 & 0 & -1 \end{vmatrix}$ = 4

And $\Delta' = \begin{vmatrix} 5 & 0 & 2 \\ -3 & 2 & 0 \\ 1 & -3 & -1 \end{vmatrix} = 4$

Therefore, we can say that for a square matrix A; *det* A = *det* A′ where A′ is the transpose of matrix A

iv) If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes

Consider $|A| = \begin{vmatrix} 1 & 3 & 5 \\ -2 & -4 & -2 \\ 4 & 2 & 4 \end{vmatrix} = 42$

If we interchange R_1 with R_3, then $\begin{vmatrix} 4 & 2 & 4 \\ -2 & -4 & -2 \\ 1 & 3 & 5 \end{vmatrix} = -42 = -\ |A|$

The interchange of two rows, say i^{th} and j^{th} is denoted as $R_i \leftrightarrow R_j$ and similarly interchange of i^{th} and j^{th}columns is denoted as $C_i \leftrightarrow C_j$

v) If two rows R_i and R_j $(or\ C_i$ and $C_j)$ of a determinant are identical, then the value of determinant is zero.

For example $\Delta = \begin{vmatrix} 5 & -3 & 1 \\ 1 & 2 & -3 \\ 1 & 2 & -3 \end{vmatrix}$

Here $R_2 = R_3 \Rightarrow \Delta = 0$

vi) If corresponding elements of any two rows R_i and R_j $(or\ columns\ C_i$ and $C_j)$ of a determinant are in the same ratio, then the determinant value is zero.

For example, $\Delta = \begin{vmatrix} -3 & 4 & -6 \\ 11 & 2 & 22 \\ 2 & 1 & 4 \end{vmatrix} = 0$ as $\frac{C_1}{C_2} = \frac{1}{2}$

vii) If all the elements in any one row R_i (or a column C_j) of the determinant are zero, then the determinant value is zero.

For example $\Delta_1 = \begin{vmatrix} 2 & 0 & 5 \\ 6 & 0 & 4 \\ 1 & 0 & -7 \end{vmatrix} = 0$ and $\Delta_2 = \begin{vmatrix} 0 & 0 \\ 4 & -9 \end{vmatrix} = 0$

viii) If some or all elements of a row or column of a determinant are expressed as sum of two (or more) terms, then the determinant can be expressed as sum of two or more determinants

For example, to evaluate $\Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

We will split R_2 as follows:

$$\Delta = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$$

Let $\Delta_1 = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix}$ *and* $\Delta_2 = \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$

As $R_1 = R_2$ in Δ_1 and $R_2 = 2R_1$ in Δ_2

Therefore, $\Delta_1 = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = 0$ *and* $\Delta_2 = \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix} = 0.$ Hence $\Delta = 0$

ix) If each element of *a row* (or a column) of a determinant is multiplied by a non-zero constant k, then the determinant value gets multiplied by k

For example, For A = $\begin{bmatrix} 3 & 5 \\ -4 & 6 \end{bmatrix}$

|A| = 18 +20 = 38

If we multiple R$_1$ by 2, we get

$\begin{vmatrix} 6 & 10 \\ -4 & 6 \end{vmatrix}$ = 36 +40 = 76 = 2|A|

Therefore, we can say when we apply $R_i \to kR_i$ (or $C_i \to kC_i$) to a determinant Δ we get $k\Delta$

What would happen if we multiply *all* rows (or columns) of |A| by 4, we get

|4A| = $\begin{vmatrix} 12 & 20 \\ -16 & 24 \end{vmatrix}$ = 288 + 320 = 608 = 4^2. 38 = 4^2 |A|

⇒ For any non-zero scalar k, k |A| = k^n |A|, where n is the order of the determinant

x) If a row R_i (or a column C_i) is added to equimultiples of corresponding elements of another row R_j (or a column C_i), then the value of determinant remains the same.

For example, If Δ = $\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$ = 2(1-8) -7(1-10) +1(8-10) =-14+63-2=47

Then applying operation $R_1 \to R_1 + 2R_2$ gives

Δ' = $\begin{vmatrix} 4 & 9 & 3 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$ = 4(1-8)-9(1-10) +3(8-10) =-28 + 81 -6 = 47

Or applying operation $C_2 \to C_2 - C_3$ gives

Δ'' = $\begin{vmatrix} 2 & 6 & 1 \\ 1 & 0 & 1 \\ 10 & 7 & 1 \end{vmatrix}$ = 2(0-7) – 6(1-10) + 1(7-0) = -14 +54 + 7 = 47

Therefore, the value of the determinant remains same if we apply the operation $R_i \to R_i + kR_j$ (or $C_i \to C_i + kC_j$) for a nonzero scalar k

If more than one operation like $R_i + kR_j$ (or $C_i \to C_i + kC_j$) is applied in one step, it should be seen that the row (or column) that is affected in one operation should not be used in another operation

Example 27

Evaluate $\Delta = \begin{vmatrix} 42 & 2 & 5 \\ 79 & 7 & 9 \\ 29 & 5 & 3 \end{vmatrix}$

Solution: Applying $C_1 \to C_1 - 8C_3$

Then $\Delta = \begin{vmatrix} 2 & 2 & 5 \\ 7 & 7 & 9 \\ 5 & 5 & 3 \end{vmatrix}$

$\Rightarrow \Delta = 0$ as $C_1 = C_2$

Example 28

Evaluate $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix}$

Solution: Applying $C_3 \to C_2 + C_3$

Then $\Delta = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix}$

$\Rightarrow \Delta = 0$ as $\frac{C_2}{C_3} = \frac{1}{a+b+c}$

Example 29

Given that a, b and c are in A.P., evaluate

$$\Delta = \begin{vmatrix} 2y+4 & 5y+7 & 8y+a \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$$

Solution: As a, b and c are in A.P. $\Rightarrow 2b = a + c$ ———— (i)

Now Applying $R_1 \to R_1 + R_3 - 2R_2$

$\Delta = \begin{vmatrix} 0 & 0 & a+c-2b \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ 3y+5 & 6y+8 & 9y+b \\ 4y+6 & 7y+9 & 10y+c \end{vmatrix}$ ———— from (i)

= 0

Example 30

Without expanding prove that

$$\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$$

Solution: Taking x common from C_1 and y common from C_2 and z common from C_3 , we get

$$\Delta = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$ we get

$$\Delta = xyz \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

Taking (x-y) common from C_1 and (y-z) common from C_2, we get

$$\Delta = xyz(x-y)(y-z) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x+y & y+z & z^2 \end{vmatrix}$$

Applying $C_2 \rightarrow C_2 - C_1$, we get

$$\Delta = xyz(x-y)(y-z) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & z \\ x+y & z-x & z^2 \end{vmatrix}$$

Taking (z-x) common from C_2, we get

$$\Delta = xyz(x-y)(y-z)(z-x) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & z \\ x+y & 1 & z^2 \end{vmatrix}$$

Expanding C_2, we get

$$\Delta = xyz(x-y)(y-z)(z-x)[0+0-1(0-1)]$$

$$= xyz(x-y)(y-z)(z-x)$$

Example 31

Without expanding, evaluate

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3$$

Solution: Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$ we get

$$\Delta = \begin{vmatrix} (b+c)^2 - a^2 & 0 & a^2 \\ 0 & (c+a)^2 - b^2 & b^2 \\ c^2 - (a+b)^2 & c^2 - (a+b)^2 & (a+b)^2 \end{vmatrix}$$

Taking a + b + c common from $C_1 and\ C_2$, we get

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1 - R_2$

$$\Delta = (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

Taking 2 common from R_3 , we get

$$\Delta = 2(a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -b & -a & ab \end{vmatrix}$$

Expanding C_1, we get

$\Delta = 2b(a+b+c)^2[(b+c-a)(abc + a^2b - ab^2 + ab^2) + 0 + a^2(0 + bc + ab - b^2)]$

$= 2abc(a+b+c)^3$

Example 32

Evaluate without expanding

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}) = abc + ab + bc + ac$$

Solution: Apply $R_1 \to \frac{R_1}{a}$, $R_2 \to \frac{R_2}{b}$ and $R_3 \to \frac{R_3}{c}$, we get

$$\Delta = abc \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & 1+\frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

Taking $1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ **common from** C_1, we get

$$\Delta = abc(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & 1+\frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - R_3$, we get

$$\Delta = abc(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})\begin{vmatrix} 0 & 0 & -1 \\ 1 & 1+\frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b} & 1+\frac{1}{c} \end{vmatrix}$$

Expanding R_1, we get

$$\Delta = abc\left(1+\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}\right)\left[0+0-1\left(\tfrac{1}{b}-1-\tfrac{1}{b}\right)\right] = abc\left(1+\tfrac{1}{a}+\tfrac{1}{b}+\tfrac{1}{c}\right)$$

$$= abc+ab+bc+ac$$

2.6.6 CHECK YOUR PROGRESS

EXERCISE–C

1. Evaluate the following

i) $\begin{vmatrix} 4 & -2 \\ 6 & -3 \end{vmatrix}$

ii) $\begin{vmatrix} 7 & 1 \\ 4 & -7 \end{vmatrix}$

iii) $\begin{vmatrix} 3 & 1 \\ -1 & 8 \end{vmatrix}$

iv) $\begin{vmatrix} -3 & -1 & 2 \\ 5 & 7 & -8 \\ -2 & -6 & 6 \end{vmatrix}$

v) $\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix}$

2. Find the area of the triangle with vertices (-2,-3), (-1,-8) and (3, 2).
3. For what value of “k” the points (k, 7), (-4, 5) and (1, -5) are collinear.
4. Represent the given matrices as the sum of a symmetric and skew symmetric matrices

i. $\begin{bmatrix} 3 & 1 \\ -1 & 8 \end{bmatrix}$

ii. $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

iii. $\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

5. Evaluate using properties of determinants:

i. $\begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$

ii. $\begin{vmatrix} x+y & y+z & x+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

iii. $\begin{vmatrix} x+y & y+z & x+z \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$

iv. $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

v. $\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

6. Prove the following using properties of determinants:

i. $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

ii. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

iii. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

iv. $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

v. $\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 1+3p+2q \\ 3 & 6+3p & 1+6p+3q \end{vmatrix} = 1$

7. Find adjoint A if

i. $A = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}$

ii. $A = \begin{bmatrix} -52 & 11 \\ 0 & 51 \end{bmatrix}$

iii. $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$

2.7 INVERSE OF A MATRIX

Consider $A = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{5}{13} & \frac{-1}{13} \\ \frac{3}{13} & \frac{2}{13} \end{bmatrix}$

Here, $AB = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} \frac{5}{13} & \frac{-1}{13} \\ \frac{3}{13} & \frac{2}{13} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

Definition: For a given square matrix A of order n, if there exists another square matrix B of the same order n, such that AB = BA = I, then A is said to be invertible and B is called the inverse matrix of A and denoted by A^{-1}

In the example above, B = A^{-1} = $\begin{bmatrix} \frac{5}{13} & \frac{-1}{13} \\ \frac{3}{13} & \frac{2}{13} \end{bmatrix}$

Properties of inverse of a matrix:

i. Inverse of a matrix, if it exists, is unique

ii. For two invertible matrices of same order, say A and B, then $(AB)^{-1} = B^{-1} A^{-1}$

iii. For an invertible matrix A, $(A^{-1})^{-1} = A$

iv. For an invertible matrix A, $(A^T)^{-1} = (A^{-1})^T$

There are many methods by which we can find inverse of a given square matrix.

i. Elementary operation (transformation) method

ii. Matrix method/ Adjugate or Adjoint method

iii. Row reduction method

2.7.1 FINDING INVERSE MATRIX BY ELEMENTARY OPERATIONS (TRANSFORMATION)

We can apply certain operations (transformations) on a matrix on rows or columns, which are known as elementary operations or transformations of a matrix

i) Any two rows (or columns) can be interchanged and denoted by $R_i \leftrightarrow R_j$ for an interchange between $i^{th}$and j^{th} rows (or $C_i \leftrightarrow C_j$ for interchange of $i^{th}$and j^{th}columns)

ii) A row (or column) can be multiplied by a non-zero scalar k, and denoted by $R_i \to kR_i$ (or $C_i \to kC_i$)

iii) A row (or column) can be added to the equi-multiples of corresponding elements of any other row or column and denoted by $R_i \to R_i + kR_j$ (or $C_i \to C_i + kC_j$) for a non-zero scalar k.

For an invertible square matrix A (i.e., A^{-1} exists)

i) Write A = IA to apply a sequence of row operations till we get, I = BA, where I is identity matrix of same order.

ii) Similarly, we write A = AI and apply a sequence of column operations till we get, I = AB, where I is identity matrix of same order.

In both the cases, the matrix B is the inverse of A.

iii) In case, after applying one or more elementary row (column) operations on A = I A (or A = A I), if we obtain all zeros in one or more rows of the matrix A on L.H.S. of the equation, then A^{-1} does not exist, or A is not invertible

Example 33

By using elementary row transformations, find inverse of matrix A = $\begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix}$

Solution: To apply row transformations, we will consider:

A = IA

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

applying $R_2 \to R_2 + 3R_1$t

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} A$$

Applying $R_2 \to \frac{R_2}{5}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix} A$$

As I = BA, where B = $\begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix} \Rightarrow$ A^{-1}= $\begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

Alternatively, we can also apply column transformations:

A = AI

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

applying $C_2 \to \frac{C_2}{5}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ \frac{-3}{5} & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

Applying $C_1 \to C_1 + \frac{3}{5} C_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

As I = AB, where B = $\begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix} \Rightarrow$ A^{-1} = $\begin{bmatrix} 1 & 0 \\ \frac{3}{5} & \frac{1}{5} \end{bmatrix}$

Example 34

Find the inverse of matrix A using elementary row operations where A = $\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$

Solution: Using row transformations: A = IA

$$\Rightarrow \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we get

$$\Rightarrow \begin{bmatrix} 2 & 3 & 10 \\ 0 & -12 & -15 \\ 0 & 0 & -50 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to \frac{R_1}{2}$ and $R_2 \to \frac{R_2}{-12}$, we get

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & 5 \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & -50 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{-1}{12} & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - \frac{3}{2}R_2$, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{25}{8} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & -50 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{6} & \frac{-1}{12} & 0 \\ -3 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to \frac{R_3}{-50}$, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{25}{8} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{8} & 0 \\ \frac{1}{6} & -\frac{1}{12} & 0 \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix} A$$

Applying $R_1 \to R_1 - \frac{25}{8}R_3$ and $R_2 \to R_2 - \frac{5}{4}R_3$, we get

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{120} & \frac{-1}{12} & \frac{1}{40} \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix} A$$

As I = AB, where B = $\begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{120} & \frac{-1}{12} & \frac{1}{40} \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{120} & \frac{-1}{12} & \frac{1}{40} \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix}$

Example 35

Find A^{-1}, if A = $\begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix}$

Solution: Using column transformations:

A = AI

$$\Rightarrow \begin{bmatrix} 10 & -2 \\ -5 & 1 \end{bmatrix} = A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Applying $C_2 \rightarrow C_2 + \frac{1}{5}C_1$, we get

$$\Rightarrow \begin{bmatrix} 10 & 0 \\ -5 & 0 \end{bmatrix} = A\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On LHS, the elements of second column are all zeroes, therefore, A^{-1} does not exist

2.7.2 FINDING INVERSE MATRIX BY INVERSE OF COEFFICIENT MATRIX METHOD

Recall that for a given square matrix A of order n, we have

$A\,(adj\ A) = (adj\ A)\,A = |A|\ I$, where I is the identity matrix of order n

$\Rightarrow A^{-1} = \frac{1}{|A|}(adj\ A)$,

Clearly $|A| \neq 0 \Rightarrow$ only a non-singular square matrix is invertible

recall that the determinant value of a singular matrix is always zero

Now let us learn how to use matrix method to find inverse of a non-singular matrix i.e., we can find inverse of a matrix A only when $|A| \neq 0$

Example 36

Find the inverse of matrix A using inverse of the coefficient matrix method when

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$$

Solution: $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$

Expanding a row or a column, we will first find $|A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 1200 \neq 0$

$\Rightarrow A^{-1}$ exists

$adj\ A$ = transpose of cofactor matrix = $\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

$\Rightarrow$ inverse of A= $A^{-1} = \frac{1}{|A|}(adj\,A) = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{16} & \frac{1}{8} & \frac{1}{16} \\ \frac{1}{120} & \frac{-1}{12} & \frac{1}{40} \\ \frac{3}{50} & 0 & \frac{-1}{50} \end{bmatrix}$$

2.8 SOLVING SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS

2.8.1 INVERSE OF COEFFICIENT MATRIX

This method is used to solve a system of equations in two or three variables.

Let us understand how to find inverse of a given square matrix A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of order 2×2

Then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Example 37

Solve the following system of equation finding the inverse of coefficient matrix:

$$x - y = 5; \ 2x + 3y = -1$$

Solution: Let us convert given equation in matrix form AX=B.

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

Here A = $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ is called coefficient matrix

And B = $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$ is called the constant matrix

Let us find inverse of coefficient matrix = $A^{-1} = \frac{1}{1\times3-(-1)\times2}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

Now $\quad AX = B \Rightarrow X = A^{-1}B$

$$\Rightarrow X = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 15-1 \\ -10-1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} \frac{14}{5} \\ \frac{-11}{5} \end{bmatrix}$$

Therefore $x = \frac{14}{5}$ and $y = \frac{-11}{5}$

Recall representing a system of linear equation in matrix formation:

Example 38

Solve for x, y and z:

$2x - 3y = 5; \quad 5x + 3y = 2$

Solution: Let us write the given system of equations in matrix form

$$\begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ so that AX = B

In this representation, matrix A is called coefficient matrix and B is called constant matrix. **Now pre-multiplying both sides by A^{-1}**, we get

$$A^{-1}(AX) = A^{-1} B$$

$$\Rightarrow I X = A^{-1} B$$

$$\Rightarrow X = A^{-1} B$$

For $A = \begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}$

As $|A| = 6 + 15 = 21 \neq 0 \Rightarrow A^{-1}$ exists and $A^{-1} = \frac{1}{|A|}(adj\,A)$

Here $adj\ A = \begin{bmatrix} 3 & 3 \\ -5 & 2 \end{bmatrix}$

Therefore $A^{-1} = \frac{1}{21}\begin{bmatrix} 3 & 3 \\ -5 & 2 \end{bmatrix}$

As $\begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 3 & 3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 15+6 \\ -25+4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{21}\begin{bmatrix} 21 \\ -21 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Hence x = 1 and y = -1

Example 39

Solve the following system of equation using matrix method

$x + y + z = 10$, $2x + y = 13$, $x + y - 4z = 0$

Solution: The given equation in matrix form can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$

As $|A| = 5 \neq 0 \Rightarrow A^{-1}$ exists

$$Adj A = \begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

Therefore $A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{5}\begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}$

Hence the matrix equation AX=B implies

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & 5 & -1 \\ 8 & -5 & 2 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 \\ 15 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

Therefore $x = 5$, $y = 3$ and $z = 2$

2.8.2 CRAMER'S RULE

Steps to Explain Cramer's rule. Cramer's rule is another method to find the solution of system of linear equations.

Example 40

Solve the following system of equation using Cramer's rule

$2x - 3y = 5;\ 5x + 3y = 2$

Solution: Let us convert given equation in matrix form AX=B.

$$\begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Here matrix $A = \begin{bmatrix} 2 & -3 \\ 5 & 3 \end{bmatrix}$ is called coefficient matrix

and $B = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is called constant matrix.

Firstly, we find the determinant of coefficient matrix called as delta (Δ).

Here $\Delta = |A| = \begin{vmatrix} 2 & -3 \\ 5 & 3 \end{vmatrix} = 6-(-15) = 21 \neq 0$

Now create a new determinant Δx by replacing "x" column by the elements of the constant matrix B

$\Rightarrow \Delta x = \begin{vmatrix} 5 & -3 \\ 2 & 3 \end{vmatrix} = 15+6 = 21$

Similarly, create another determinant Δy by replacing "y" column

$\Delta y = \begin{vmatrix} 2 & 5 \\ 5 & 2 \end{vmatrix} = 4 - 25 = -21$

For a system of equations
$a_1 x + b_1 y = 0$
and, $a_2 x + b_2 y = 0$
as the elements of constant matrix are zero, i.e. B =
Such system of equations are called *homogenous* equations

Finally x is obtained by dividing Δx by Δ, and Δy by Δ

$$\Rightarrow x = \frac{\Delta x}{\Delta} = \frac{21}{21} = 1$$

And $y = \frac{\Delta y}{\Delta} = -\frac{21}{21} = -1$

Therefore x = 1 and y = -1

In Cramer's rule, when $\Delta \neq 0$, then the given system of equations

- is consistent
- has a unique solution.

Example 41

Solve the following system of equations using Cramer's rule

x + y + z = 10 , 2x + y = 13 , x + y - 4z = 0

Solution: Convert the given equations in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 10 \\ 13 \\ 0 \end{bmatrix}$

Here $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & -4 \end{vmatrix} = 1\,[-4-0] - 1\,[-8-0] + 1\,[2-1] = 5 \neq 0$

Now $\Delta x = \begin{vmatrix} 10 & 1 & 1 \\ 13 & 1 & 0 \\ 0 & 1 & -4 \end{vmatrix} = 10\,[-4] -1\,[-52] + 1[13] = 25$

$\Delta y = \begin{vmatrix} 1 & 10 & 1 \\ 2 & 13 & 0 \\ 1 & 0 & -4 \end{vmatrix} = 15$

And, $\Delta z = \begin{vmatrix} 1 & 1 & 10 \\ 2 & 1 & 13 \\ 1 & 1 & 0 \end{vmatrix} = 10$

$$\Rightarrow x = \frac{\Delta x}{\Delta} = \frac{25}{5} = 5,$$

$$\Rightarrow y = \frac{\Delta y}{\Delta} = \frac{15}{5} = 3$$

and $z = \frac{\Delta z}{\Delta} = \frac{10}{5} = 2$

Therefore, x = 5, y =3 and z = 2

Example 42

Solve the following system of equations using Cramer's rule,

2x – 3y = 5; -4x + 6y = -10

Solution: Let us convert given equations in matrix form AX=B.

$$\begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

Here $\Delta = \begin{vmatrix} 2 & -3 \\ -4 & 6 \end{vmatrix} = 0$

$\Delta x = \begin{vmatrix} 5 & -3 \\ -10 & 6 \end{vmatrix} = 0$

$\Delta y = \begin{vmatrix} 2 & 5 \\ -4 & -10 \end{vmatrix} = 0$

The given system of equation has infinitely many solutions.

In Cramer's rule, when $\Delta = 0$ and all other delta values Δx, Δy and Δz are zero; then the system of equations

- Is dependable consistent.
- has infinitely many solutions.

Example 43

Solve the following system of equations using Cramer's rule

$x - 2y + 3z = 1$, $2x + y - z = 3$ and $3x - y + 2z = -2$

Solution: Convert given equations in matrix form

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 1\,[2-1] + 2\,[4+3] + 3\,[-2-3] = 0$$

$$\Delta x = \begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & -1 \\ -2 & -1 & 2 \end{vmatrix} = 1[2-1] + 2\,[6-2] + 3\,[-3+2] = 6 \neq 0$$

$$\Delta y = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & -1 \\ 3 & -2 & 2 \end{vmatrix} = 1(6-2)-1(4+3) +3(-4-9) = 4 - 7 -39 = -42 \neq 0$$

$$\Delta z = \begin{vmatrix} 1 & -2 & 1 \\ 2 & 1 & 3 \\ 3 & -1 & -2 \end{vmatrix} = 1(-2+3) +2(-4-9) +1(-2-3) = 1 - 26 - 5 = -30 \neq 0$$

Here $\Delta = 0$ and the other deltas Δx, Δy and Δz are non-zero values. In such a case the given system of equations is inconsistent angel has no solution.

In Cramer's rule, when $\Delta = 0$ and any one or more of the deltas Δx, Δy or Δz are non-zero then the given system of equations

- is inconsistent
- has no solution.

2.8.3 ROW REDUCTION METHOD

Example 44

Solve using row reduction method: $x - 3y = 9$, $2x + y + 3 = 0$

Solution: Write all the given equations in the form $ax + by = c$

$$\Rightarrow x - 3y = 9$$
$$2x + y = -3$$

Now we shall write the given system of equations in the augmented matrix as shown below:

$$\begin{array}{l} \text{Equation (i)} \longrightarrow \\ \text{Equation (ii)} \longrightarrow \end{array} \begin{bmatrix} 1 & -3 & 9 \\ 2 & 1 & -3 \end{bmatrix}$$

In this method, we will reduce Row_1 by making $a_{11} = 1$ and the rest of the elements in $Column_1$ as 0 by using the elementary row transformations

$$R_2 \to R_2 - 2R_1 \Rightarrow \begin{bmatrix} 1 & -3 & 9 \\ 0 & 7 & -21 \end{bmatrix}$$

Then we shall rewrite the system of equations, using the above row reduced matrix:

$R_1 \to$ x -3y = 9 …..(i)

And $R_2 \to$ 0x + 7y = - 21 $\Rightarrow y = -3$

Substituting value of y in equation (i), we get:

x -3(-3) = 9 $\Rightarrow$ x = 0

Example 45

Use row reduction method to solve the given system of equations:

3x +2y – z = 1

X + 2y – 2z = 0

And 2x +y -3z = -1

Solution: Let 3x +2y – z = 1 …… (i)

x + 2y - 2z = 0 ……(ii)

And 2x +y -3z = -1 ….. (iii)

Write the given system of equation in the augmented matrix as

$$\begin{array}{l} \text{Equation (i)} \longrightarrow \\ \text{Equation (ii)} \longrightarrow \\ \text{Equation (iii)} \longrightarrow \end{array} \begin{bmatrix} 3 & 2 & -1 & 1 \\ 1 & 2 & -2 & 0 \\ 2 & 1 & -3 & -1 \end{bmatrix}$$

We will begin by converting $a_{11} = 1$ and the rest of the elements in $Column_1$ as 0 by using the elementary row transformations

$$R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 3 & 2 & -1 & 1 \\ 2 & 1 & -3 & -1 \end{bmatrix}$$

$$R_2 \to R_2 - 3R_1 \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -4 & 5 & 1 \\ 2 & 1 & -3 & -1 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_1 \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -4 & 5 & 1 \\ 0 & -3 & 1 & -1 \end{bmatrix}$$

$$R_2 \to \frac{R_2}{-4} \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & \frac{-5}{4} & \frac{-1}{4} \\ 0 & -3 & 1 & -1 \end{bmatrix}$$

$$R_2 \to R_3 + 3R_2 \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & \frac{-5}{4} & \frac{-1}{4} \\ 0 & 0 & \frac{-11}{4} & \frac{-7}{4} \end{bmatrix}$$

$$R_3 \to \frac{-4R_3}{11} \Rightarrow \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & \frac{-5}{4} & \frac{-1}{4} \\ 0 & 0 & 1 & \frac{7}{11} \end{bmatrix}$$

Now we shall write the above matrix into a system of equations one more time:

$x + 2y - 2z = 0$

$y - \frac{5}{4}z = \frac{-1}{4}$

and $z = \frac{7}{11}$

$\Rightarrow y - \frac{5}{4} \times \frac{7}{11} = \frac{-1}{4} \Rightarrow y = \frac{6}{11}$

Also $x + 2\times\frac{6}{11} - 2\times\frac{7}{11} = 0 \Rightarrow x = \frac{2}{11}$

Therefore $x = \frac{2}{11}$, $y = \frac{6}{11}$ and $z = \frac{7}{11}$

Example 46

Three shopkeepers A, B and C are using polythene bags, handmade bags and newspaper bags. A uses 20, 30 and 40 number of bags of respective type. B uses 30, 40 and 20 of each respective kind while C uses 40, 20 and 30 of each type. Each shopkeeper spent Rs 250, Rs 220 and Rs 200 on the bags. Find the cost of each carry bag using matrix method.

Solution:

Shopkeeper	Polythene bags	Handmade bags	Newspaper bags	Total cost (in ₹)	Cost per bag
A	20	30	40	250	x
B	30	40	20	220	y
C	40	20	30	200	z

As per the question,

$20x + 30y + 40z = 250 \Rightarrow 2x + 3y + 4z = 25$

$30x + 40y + 20z = 220 \Rightarrow 3x + 4y + 2z = 22$

and, $40x + 20y + 30z = 200 \Rightarrow 4x + 2y + 3z = 20$

Writing in matrix form: $\begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 25 \\ 22 \\ 20 \end{bmatrix}$

Where $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

As $|A| = 2(12-4) - 3(9-8) + 4(6-16) = 16 - 3 - 40 = -27 \neq 0 \Rightarrow A^{-1}$ exists

$$Adj\ A = \begin{bmatrix} 8 & -1 & -10 \\ -1 & -10 & 8 \\ -10 & 8 & -1 \end{bmatrix}' = \begin{bmatrix} 8 & -1 & -10 \\ -1 & -10 & 8 \\ -10 & 8 & -1 \end{bmatrix}$$

$\therefore \quad A^{-1} = \frac{1}{|A|}(adj\,A) = \frac{-1}{27}\begin{bmatrix} 8 & -1 & -10 \\ -1 & -10 & 8 \\ -10 & 8 & -1 \end{bmatrix}$

As AX = B then $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{27}\begin{bmatrix} 8 & -1 & -10 \\ -1 & -10 & 8 \\ -10 & 8 & -1 \end{bmatrix}\begin{bmatrix} 25 \\ 22 \\ 20 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{27}\begin{bmatrix} 25 \\ 22 \\ 20 \end{bmatrix} = \begin{bmatrix} \frac{22}{27} \\ \frac{85}{27} \\ \frac{94}{27} \end{bmatrix}$

Therefore cost of a polythene bag = ₹ $\frac{22}{27}$

cost of a handmade bag = ₹ $\frac{85}{27}$

cost of a newspaper bag = ₹ $\frac{94}{27}$

Example 47

A school plans to award ₹ 6000 in total to its students to reward for certain values - honesty, regularity and hard work. When three times the award money for hard work is added to the award money given for honesty amounts to ₹ 11000. The award money for honesty and hard work together is double the award money for regularity. Use matrix method to find the prize money for each category of award.

Solution: Let the prize money for honesty = ₹ x

Prize money for regularity = ₹ y

And prize money for hard work = ₹ z

As per the question. $x + y + z = 6000$

$x + 3z = 11000$

and $x + z = 2y \Rightarrow x - 2y + z = 0$

Writing in matrix form: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & -2 & 1 \end{bmatrix}$

As $|A| = 1(0+6) - 1(1-3) + 1(-2+0) = 6 + 2 - 2 = 6 \neq 0 \Rightarrow A^{-1}$ exists

$adj\ A = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}' = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$

$\therefore \quad A^{-1} = \frac{1}{|A|}(adj\, A) = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$

As AX = B then X = A^{-1} B $\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$

Therefore, prize money for each prize for honesty = ₹ 500

Prize money for each prize for regularity = ₹ 2000

Prize money for each prize for hard work = ₹ 3500

2.8.4 CHECK YOUR PROGRESS

EXERCISE – D

1. Find the inverse of the given matrices, by using elementary transformations:

   i) $\begin{bmatrix} -5 & -1 \\ 3 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$

   iii) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ iv) $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

2. Find inverse of the given matrices, by using Adjugate (matrix) method:

   i. $\begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$ *ii.* $\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$

3. Solve the following system of equations by i) Matrix method ii) Row reduction method:
   a) $2x - 3y = -4, \; 3x + 5y = 13$
   b) $x + y = 1, \; 5x - 7y = 29$
   c) $5x - 4y = 9, \; 3x + 7y = -4$
   d) $x - y + 2z = 1, \; 2y - 3z = 1, \; 3x - 2y + 4z = 2$
   e) $2x - 3y + 5z = 1, \; 3x + 2y - 4z = -5, \; x + y - 2z = -3$
4. Solve the following system of equation using Cramer's rule.
   i) $2x - 3y = -4, \; 3x + 5y = 13$
   ii) $x + y = 1, \; 5x - 7y = 29$
   iii) $5x - 4y = 9, \; 3x + 7y = -4$
   iv) $x - 3y = 4, \; 3x - 9y = 12$
   v) $-2x + y = 3, \; 4x - 2y = 5$
   vi) $x - y + 2z = 1, \; 2y - 3z = 1, \; 3x - 2y + 4z = 2$
   vii) $2x - 3y + 5z = 1, \; 3x + 2y - 4z = -5, \; x + y - 2z = -3$
   viii) $x + y + z = 0, \; -3x + y - 4z = 0, \; -2x + 2y - 3z = 0$
   ix) $2x - y - 3z = 1, \; 3x + 2y - 5z = 0, \; 5x + y - 8z = 3$

2.9 APPLICATION OF MATRICES AND DETERMINANTS

Matrices and determinants are powerful tools in modern mathematics, which have a wide range of application. Sociologists use matrices to study the dominance within a social group or society. Demographers make use of matrices to study survival of mankind, marriage and decent structure. Business mathematics, economist, artificial intelligence coding and networking models are a few examples that function on the concept of matrix and determinant. The study of communication theory and electrical engineering as the network analysis is also aided by the use of matrix representation.

2.7.1 Leontief input-output model for two variables

The economy of any country is dependent on many sectors which are interlinked with each other. In this section we try to learn whether the interlinks are viable or not. If the system is viable then try to find the interlink inputs based on the demand in the market. Many sectors are sharing their resources and try to become self-sufficient and independent.

To understand the technique, let's consider two sectors, Automobile Sector (AS) and Electrical Sector (ES). Both are dependent on each other. As part of the resources are used by one industry and remaining resources are shared with other sectors.

Consider the situation based on the given information, that automobile sector uses 20 units of resources for themselves and share 15 units to electrical sector. The electrical sector is using 30 units of resources for themselves and share 12 units with the automobile sector.

As many sectors share their respective resources with each other, In the given situation let the total resources produced by AS be 60 and ES as 45

Let us represent this situation in the following manner:

	AS	ES	—	——	——	Total
Resources produced by AS	20	15	—	—	—	60
Resources produced by ES	12	30	—	—	——	45

But make sure that no column additions are possible as the units are different.

	AS	ES	—	——	——	Total
Resources produced by AS	20/60	15/45	—	—	—	60
Resources produced by ES	12/60	30/45	—	—	——	45

Now we try to find per unit worth of output

This representation is called input-output coefficient unit matrix.

$$\text{I/0 Coefficient unit Matrix} = \begin{bmatrix} \frac{20}{60} & \frac{15}{45} \\ \frac{12}{60} & \frac{30}{45} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{2}{3} \end{bmatrix}$$

Now let us introduce one more additional information that every sector is in demand at a particular time, which may change with respect to time.

So the demand matrix is always changing accordingly there is impact on coefficient matrix.

Let us consider the demand as follows

	AS	ES	-	-	—	Total	Demand
Resources produced by AS	20	15	-	-	-	60	70
Resources produced by ES	12	30	-	-	—	45	50

Now we shall convert problem into matrix form

$$AX + D = X \quad \ldots\ldots\ldots(1)$$

where A is an input-output unit matrix(technology matrix).

D is the total demand

X is the new requirement output.

From equation (1) $\qquad D = I.X - A.X$

$\Rightarrow \qquad D = (I - A)\,X$

Therefore, the requirement output $X = (I\text{-}A)^{-1}D$

Hawkins-Simon Conditions to check for **viability** of an economy are defined as:

i) $|\ I - A\ |$ must be positive

ii) Diagonal element of I - A must be positive.

Let us take an example to test the above stated conditions

Here, $A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{2}{3} \end{bmatrix}$

then $I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{5} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{5} & \frac{1}{3} \end{bmatrix}$

$\Rightarrow |I\text{-}A| = \begin{vmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-1}{5} & \frac{1}{3} \end{vmatrix} = \frac{2}{9} - \frac{1}{15} = \frac{7}{45} > 0$ which verifies the first Hawkins-Simon condition

Also main diagonal of I-A is positive.

$\Rightarrow$both of the Hawkins-Simon conditions are satisfied.

Hence required input to fulfil the demand and is viable

The requirement output $X = (I\text{-}A)^{-1}D$

$\Rightarrow X = \frac{1}{|I-A|} adj(I - A).D$

In the above example,

$$X = \frac{1}{7/45}\begin{bmatrix} 1/3 & 1/3 \\ 1/5 & 2/3 \end{bmatrix}\begin{bmatrix} 70 \\ 50 \end{bmatrix}$$

$$X = \begin{bmatrix} 5400/21 \\ 31950/135 \end{bmatrix}$$

> Hawkins-Simon Condition
> - $|I\text{-}A|$ must be positive
> - Diagonal element of I-A must be positive.

Example 48

Prepare an input-output table for Transport industry (TI) and Food industry (FI). Food industry produces 50 units. Out of these 20 units consumed by FI and 25 units by TI. Whereas Transport industry produces 40 units and out of these 10 units used by FI and 25 units by TI.

Construct input- output matrix. Check the condition of Hawkins-Simon condition and decide whether system is viable. If so find the input to fulfil the demand 100 of FI and 80 of TI.

Solution:

	FI	TI	Total
Food Industry	20	25	50
Transport Industry	10	25	40

Input-output coefficient matrix (A) = $\begin{bmatrix} 20/50 & 25/40 \\ 10/50 & 25/40 \end{bmatrix} = \begin{bmatrix} 2/5 & 5/8 \\ 1/5 & 5/8 \end{bmatrix}$

Firstly we will test the viability using Hawkins-Simon condition

Get I- A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2/5 & 5/8 \\ 1/5 & 5/8 \end{bmatrix} = \begin{bmatrix} 3/5 & -5/8 \\ -1/5 & 3/8 \end{bmatrix}$

Now $|I-A| = \begin{vmatrix} 3/5 & -5/8 \\ -1/5 & 3/8 \end{vmatrix} = \frac{9}{40} - \frac{1}{8} = \frac{9-5}{40} = \frac{1}{10}$

Clearly diagonal positions of I-A are positive. System is viable.

Here demand matrix is given as = D = $\begin{bmatrix} 100 \\ 80 \end{bmatrix}$

So $\quad X = (I-A)^{-1}D$

$X = \frac{1}{|I-A|} adj(I-A).D$

$X = \frac{1}{10} \begin{bmatrix} 3/8 & 5/8 \\ 1/5 & 3/5 \end{bmatrix} \begin{bmatrix} 100 \\ 80 \end{bmatrix}$

$X = \frac{1}{10} \begin{bmatrix} \frac{300}{8} + \frac{400}{8} \\ \frac{100}{5} + \frac{240}{5} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 700/8 \\ 340/5 \end{bmatrix} = \begin{bmatrix} 70/8 \\ 34/5 \end{bmatrix}$

Hence, to fulfil the demand FI must share 70/8 units and TI must share 34/5 units.

2.9.2 CHECK YOUR PROGRESS

EXERCISE – E

1. **Solve the following problems using Leontief input-output model.**

	Sector 1	**Sector 2**	**Total**
Sector 1	2	5	10
Sector 2	3	4	20

If the system is viable then discuss the situation for new demand 8 and 12 from sector 1 and sector 2 respectively.

2. **Solve the following problems using Leontief input-output model.**

	FI	AI	Total
Food industry	20	10	40
Agricultural industry	30	20	60

If the system is viable then discuss the situation for new demand 80 and 120 from FI and AI respectively.

3. Solve the following problems using Leontief input-output model.

	Sector 1	**Sector 2**	**Total**
Sector 1	12	20	40
Sector 2	15	20	30

If the system is viable then discuss the situation for new demand 8 and 8 from sector 1 and sector 2 respectively.

4. Solve the following problems using Leontief input-output model.

	Sector 1	**Sector 2**	**Total**
Sector 1	5	7	30
Sector 2	6	14	21

If the system is viable then discuss the situation for new demand 8 and 12 from sector 1 and sector 2 respectively.

2.10 UNIT SUMMARY

1. A rectangular array or table of numbers, symbols, expressions or functions, when arranged in rows and columns, is known as a *matrix* (plural *matrices*). Each member of this arrangement is called an *element* of the matrix.

2. A matrix is expressed by using capital English alphabet, say A = $[a_{ij}]$ where a_{ij} is the element at the ith row and jth column of the matrix and $1 \leq i \leq m, 1 \leq j \leq, \forall\, i, j \epsilon N$

3. For a matrix A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \dots \dots \; a_{1n} \\ a_{21} & a_{22} & a_{23} \dots \dots . \, a_{2n} \\ a_{31} & a_{32} & a_{n3} \dots \dots a_{mn} \end{bmatrix}$ having 'm' number of rows and 'n' number of columns, the expression m x n is called the *order* of the given matrix A where $1 \leq i \leq m, 1 \leq j \leq n, \forall\, i, j \epsilon N$

   The elements $a_{11}, a_{22}\,, a_{33}, \dots$ where $a_{ij}, \forall\, i = j$ are called elements of the *diagonal* of the matrix

   And the elements where $a_{ij}, \forall i \neq j$ are called elements of the non-*diagonal* of the matrix

4. A matrix in which number of rows is not equal to number of columns is called a *Rectangular matrix*

5. A matrix in which the number of rows and columns are equal is called a *Square matrix*

6. A matrix having exactly one row is called a *Row matrix.*

7. A matrix having exactly one column is called a *Column matrix.*

8. A square matrix in which all the non-diagonal entries are zero, i.e. $a_{ij} = O, \forall\, i \neq j$ is called a *Diagonal matrix*

9. A diagonal matrix having same diagonal elements, i.e. $a_{ij} = k, \forall\, i = j$ where $k \neq 0$ is called a *Scalar matrix.*

10. A scalar matrix in which all the diagonal entries are equal to 1,
i.e. $a_{ij} = 1, \forall\, i = j$ is called an *Identity matrix (Unit matrix),* denoted by English alphabet I

11. A matrix with each of its elements as zero,
i.e. $a_{ij} = 0, \forall \leq i \leq m, 1 \leq j \leq n, \forall\, i, j \epsilon N$ is called a *zero matrix*

12. Two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$ having same order m×n are called *Equal matrices* when each element of A is equal to the corresponding element of B,
i.e. $a_{ij} = b_{ij}\ \forall\, 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

13. For a matrix A = $[a_{ij}]$ of order m × n and k is a scalar quantity, then kA is another matrix obtained by multiplying each element of A by the scalar quantity k,
i.e. kA = k $[a_{ij}]$ = $[ka_{ij}]$, $\forall\, 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

14. If A = $[a_{ij}]$ and B = $[b_{ij}]$ are two matrices of the same order, say m × n, and k and p are scalars, then
(i) k (A +B) = k A + k B
(ii) (k + p) A = k A + p A
(iii) k (A + B) = kA + kB

15. For a non-zero matrix A, of order m×n, a matrix B of same order is called *Negative matrix* of matrix A such that A + B = O, where O is the zero matrix of the same order. We denote negative matrix A as – A

16. For two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$, of same order m × n, the sum of two matrices A and B is defined as a matrix S = A+B = $[s_{ij}]$ of order m × n such that
$s_{ij} = a_{ij} + b_{ij}\ \forall\, 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

17. Addition of two or more matrices is possible only when the given matrices are of same order. The order of resultant matrix is also same
$\Rightarrow A_{m\times n} + B_{m\times n} = C_{m\times n}$
Matrix addition is commutative$\Rightarrow A_{m\times n} + B_{m\times n} = B_{m\times n} + A_{m\times n}$
Matrix addition is associative $\Rightarrow A_{m\times n} + (B_{m\times n} + C_{m\times n}) = (A_{m\times n} + B_{m\times n}) + C_{m\times n}$
Zero matrix is the additive identity $\Rightarrow A_{m\times n} + O_{m\times n} = A_{m\times n} = O_{m\times n} + A_{m\times n}$
Negative of a matrix is the additive inverse in matrix addition$\Rightarrow A_{m\times n} + (-A_{m\times n}) = O_{m\times n}$

18. For two matrices A = $[a_{ij}]$ and B = $[b_{ij}]$, of same order m × n, the difference of two matrices A and B is defined as a matrix D = A - B = $[d_{ij}]$ of order m ×n such that $d_{ij} = a_{ij} - b_{ij}$ $\forall\, 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

19. For two matrices A = $[a_{ij}]$ of order m×n and B = $[b_{jk}]$ of order n × p, the multiplication of the matrices AB is defined as a matrix P = AB = $[p_{ik}]$ of order m ×p such that for finding the p_{ik} we multiple ith row of frist matrix A with the kth column of second matrix B and calculate the sum of these products
i.e. $p_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + a_{i4}b_{4k} + \cdots + a_{in}b_{nk}\ \forall\, 1 \leq i \leq m \text{ and } 1 \leq j \leq n$

20. i. The multiplication of matrices is associative i.e. for any three matrices A, B and C, (AB) C = A (BC), whenever order of multiplication is defined on both sides

ii. Distributive property of multiplication holds true for multiplication of matrices. i.e. for three matrices A, B and C, A (B+C) = AB + AC

iii. A (B - C) = AB - AC, whenever order of multiplication is defined on both sides.

21. For a given square matrix A of order m$\times$m, there exists a multiplicative identity matrix I, of same order such that IA = AI = A.

22. For a matrix A = $[a_{ij}]$ of order m$\times$n, the matrix obtained by interchanging the rows and columns of the matrix is called the transpose of matrix A

23. For a matrix A of order m$\times$n, the order of transpose matrix A is n $\times$m

24. For given matrices A and B:

i. (A′)′ = A

ii. (kA)′ = kA′ (where k is any constant)

iii. (A + B)′ = A′ + B′

iv. (AB)′ = B′A′

24. For a given square matrix A = $[a_{ij}]$, if A′ = A, $\forall\ 1 \le i \le m \text{ and } 1 \le j \le n$, then the matrix A is called a symmetric matrix

25. For a given square matrix A = $[a_{ij}]$, if A′ = - A, $\forall\ 1 \le i \le m \text{ and } 1 \le j \le n$, then the matrix A is called a skew-symmetric matrix

26. For a square matrix A having real values as elements,

i. A + A′ is a symmetric matrix

ii. A – A′ is a skew symmetric matrix

27. Let X be a set of square matrices and R be the set of numbers (real or complex) such that a function f is defined as f : M → K by f (A) = k, where A ∈ X and k ∈ R, then f(A) is called the determinant of A

28. In a given determinant, minor of an element a_{ij} is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies, and is denoted by M_{ij}.

29. In a square matrix, cofactor of an element a_{ij} , denoted by A_{ij} or C_{ij} is defined by

$A_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor of a_{ij}

30. The transpose of cofactors matrix of a square matrix A = $[a_{ij}]$ is called Adjoint matrix and is denoted by *adj* A.

31. For a given square matrix A, of order n,

i. A(adj A) = (adj A) A = |A| I , where I is the identity matrix of order n

ii. |*adj*(A)| = $|A|^{n-1}$

32. If the determinant of any matrix is zero then that matrix is called Singular Matrix and if |A| ≠ 0 then A is called Non-singular matrix.

33. Let A(x_1,y_1), B(x_2,y_2) and C(x_3,y_3) be the vertices of the triangle ABC, then

Area $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

34. For a given square matrix A of order n, if there exists another square matrix B of the same order n, such that AB = BA = I, then A is said to be invertible and B is called the inverse matrix of A and denoted by A^{-1}

    i. Inverse of a matrix, if it exists, is unique

    ii. For two invertible matrices of same order, say A and B, then $(AB)^{-1} = B^{-1} A^{-1}$

    iii. For an invertible matrix A, $(A^{-1})^{-1} = A$

    iv. For an invertible matrix A, $(A^T)^{-1} = (A^{-1})^T$

35. A system of equations can be solved by using any of the following methods:

    i. inverse of coefficient method

    ii. Cramer's method

    iii. Row reduction method

36. Hawkins-Simon Conditions to check for **viability** of an economy are defined as:

    i. | I – A | must be positive

    ii. Diagonal elements of (I – A) must be positive

2.11. ANSWERS TO THE EXERCISES

EXERCISE – A

1. i) Row, 1×2 ii) Column, 3×1
   iii) Rectangle, 3×2 iv) Row 1×3
   v) Square 2×2 vi) Square 3×3
   vii) Zero 2×2 viii) Zero 2×3
   ix) Identity 3×3 x) Scalar 3×3
   i) Scalar 3×3

2. i) – 4
   ii) 0 + 2
   iii) 4

3. $\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32 \end{bmatrix}$

4. $\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{bmatrix}$

5. 81

6. 4

7. a = 14, b = -7, c = 7, d = 1

EXERCISE – B

1. Order of the matrix

A	B	A$\pm$B	AB
2×2	2×2	2×2	2×2
2×3	3×2	Not possible	2×2
3×4	4×1	Not possible	3×1
3×3	3×3	3×3	3×3
2×3	2×3	2×3	Not possible
1×3	3×2	Not possible	1×2

5. i) $\begin{bmatrix} 35 & -20 \\ -28 & 27 \end{bmatrix}$ ii) $[-4]$

iii) $\begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$ iv) p=1, q = 2, r = 3, s = 4

v) $\begin{bmatrix} 4 & -11 \\ -5 & -1 \end{bmatrix}$

6. $\begin{bmatrix} -5 & -1 \\ -1 & 5/2 \\ -19/4 & -4 \end{bmatrix}$

7. i. $\begin{bmatrix} -5 & -8 & 12 \\ -8 & 0 & 20 \\ -6 & 5 & -11 \end{bmatrix}$ $ii.$ $\begin{bmatrix} -5 & -8 & 12 \\ -8 & 0 & 20 \\ -6 & 5 & -11 \end{bmatrix}$

iii. $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ iv. $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

v. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

9. In the month of March P = $\begin{array}{c} \begin{bmatrix} 250 & 400 \\ 230 & 425 \end{bmatrix} \end{array} \begin{array}{l} bookseller A \\ bookseller B \end{array}$

In the month of April Q = $\begin{array}{c} \begin{bmatrix} 550 & 300 \\ 270 & 450 \end{bmatrix} \end{array} \begin{array}{l} bookseller A \\ bookseller B \end{array}$

Total Sell P+Q = $\begin{bmatrix} 250 & 400 \\ 230 & 425 \end{bmatrix} + \begin{bmatrix} 550 & 300 \\ 270 & 450 \end{bmatrix}$

$= \begin{bmatrix} 800 & 700 \\ 500 & 875 \end{bmatrix} \begin{array}{l} bookseller A \\ bookseller B \end{array}$

10. Sell matrix

$$S = \begin{array}{c} \text{pen} \quad \text{notebooks} \\ \begin{bmatrix} 5 & 7 \\ 6 & 4 \end{bmatrix} \end{array} \begin{array}{l} \\ bookseller P \\ bookseller Q \end{array}$$

Cost Matrix

$$C = \begin{bmatrix} 12 \\ 27 \end{bmatrix}$$

Amount Matrix $A = S\,C$

$$A = \begin{bmatrix} 5 & 7 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 12 \\ 27 \end{bmatrix}$$

$$= \begin{bmatrix} 60 + 189 \\ 72 + 108 \end{bmatrix}$$

$$= \begin{bmatrix} 249 \\ 180 \end{bmatrix}$$

Bookseller P generates the amount Rs 249/- and Q generates the amount Rs 180/-

EXERCISE – C

1. i) 0, ii) -53, iii)25, iv) 0, v) 18

2. 15 sq. units

3. -5

7. i. $\begin{bmatrix} 5 & -1 \\ 3 & 2 \end{bmatrix}$

ii. $\begin{bmatrix} 51 & -11 \\ 0 & -52 \end{bmatrix}$

iii. $\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -2 \\ 2 & -1 & 5 \end{bmatrix}$

EXERCISE – D

1. *i)* $\frac{1}{-7}\begin{bmatrix} 2 & 1 \\ -3 & -5 \end{bmatrix}$ *ii)* $\begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$

   iii) $\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$ iv) : $\frac{1}{5}\begin{bmatrix} 2 & 1 & -5 \\ 1 & -2 & 5 \\ -3 & 1 & 5 \end{bmatrix}$

2. *i)* $\frac{1}{11}\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ *ii)* $\begin{bmatrix} -2 & 1/2 & 1 \\ 11 & -1 & -6 \\ 4 & -1/2 & -2 \end{bmatrix}$

3. a) x =1, y =2 b) x = 3, y = -2
   c) x = 1, y = -1 d) x = 0, y = 5, z = 3
   e) x = 1, y = 2, z = 3

4. i) x = 1, y = 2 ii) x = 3, y = -2
iii) x = 1, y = -1) iv) Infinite many solution
v) No solution vi) x = 0, y = 5, z = 3
vii) x = 1 , y = 2, z = 3 viii) Infinite many solution
ix) No solution

EXERCISE – E

1. Coefficient unit matrix = $\begin{bmatrix} 2/10 & 5/20 \\ 3/10 & 4/20 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/4 \\ 3/10 & 1/5 \end{bmatrix}$

Get I-A = $\begin{vmatrix} 4/5 & -1/4 \\ -3/10 & 4/5 \end{vmatrix}$,

Check conditions det (I-A) = 16/25 – 3/40 = 109/200 > 0, main diagonal entries positive

So X = $(I\text{-}A)^{-1}D = \frac{200}{109}\begin{bmatrix} 4/5 & 1/4 \\ 3/10 & 4/5 \end{bmatrix}\begin{bmatrix} 8 \\ 12 \end{bmatrix}$

2. Coefficient unit matrix = $\begin{bmatrix} 20/40 & 10/60 \\ 30/60 & 20/60 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/6 \\ 1/2 & 1/3 \end{bmatrix}$

Get I-A = $\begin{vmatrix} 1/2 & -5/6 \\ -1/2 & 2/3 \end{vmatrix}$,

Check conditions det (I-A) = 1/6 – 5/12 = - 3/12 < 0,
First condition of Hawkins-Simon is not true. So condition is **not viable**

3. Coefficient unit matrix = $\begin{bmatrix} 12/40 & 20/30 \\ 15/40 & 20/30 \end{bmatrix} = \begin{bmatrix} 3/10 & 2/3 \\ 3/8 & 2/3 \end{bmatrix}$

Get I-A = $\begin{vmatrix} 7/10 & -2/3 \\ -3/8 & 1/3 \end{vmatrix}$,

Check conditions det (I-A) = 7/30 – 1/4 = (14-15)/60 < 0,
And main diagonal entries positive.
First condition of Hawkins-Simon is not true. So condition is **not viable**.

4. Coefficient unit matrix = $\begin{bmatrix} 5/30 & 7/21 \\ 6/30 & 14/21 \end{bmatrix} = \begin{bmatrix} 1/6 & 1/3 \\ 1/5 & 2/3 \end{bmatrix}$

Get I-A = $\begin{vmatrix} 5/6 & -1/3 \\ -1/5 & 1/3 \end{vmatrix}$,

Check conditions det (I-A) = 5/18 - 1/15 = (25-6)/90 = 19/90 > 0, main diagonal entries positive

So X = $(I\text{-}A)^{-1}D = \frac{90}{19}\begin{bmatrix} 2/3 & -1/3 \\ -1/5 & 1/6 \end{bmatrix}\begin{bmatrix} 8 \\ 12 \end{bmatrix}$

❑❑❑