D I F F E R E N T I A L   C A L C U L U S

LIMITS, FIRST PRINCIPALS, RULES OF DIFFERENTIATION AND THE EQUATION OF A TANGENT TO A FUNCTION

Calculus is an interesting branch of mathematics that deals with the rate of change. It is a truly fascinating tool with innumerable applications to "real life". Calculus is useful in finding the slope of any curve at a particular point, if it exists.

1. LIMITS

- When finding a limit, we look at what happens to a function value of a curve (y – value) as we get closer and closer to a specific x – value on the curve.
- When writing a limit, we use the notation $\lim_{x \to c} f(x)$
    - ✓ The abbreviation 'lim' tells us we are finding a limit
    - ✓ The notation '$x \to c$' tells us which specific value the x – value is approaching
    - ✓ $f(x)$ represents the function with which we are working

The limit is a useful tool in the calculus. In order to find a limit if it exists, one needs to simply substitute the *x value* into the function that *x is approaching*. The substitution is done mentally and not shown. In the case of complicated problems it is important to simplify the expression problem using factorisation and other techniques before finding the limit.

- Examples:

Evaluate the following limits:

1. $\lim_{x \to 2} 2x + 3 = 7$
2. $\lim_{x \to -1} (1 - x) = 2$
3. $\lim_{x \to -1} (x^2 + 2x + 1) = 0$
4. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3} (x + 3) = 6$
5. $\lim_{x \to 2} \frac{8 - x^3}{2 - x} = \lim_{x \to 2} \frac{(2-x)(4+2x+x^2)}{2-x} = \lim_{x \to 2} (4 + 2x + x^2) = 12$
6. $\lim_{h \to 0} (4 + h) = 4$
7. $\lim_{h \to 0} \frac{(2+h)^2 - 2^2}{h} = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h} = \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} \frac{h(4+h)}{h} = \lim_{h \to 0} (4 + h) = 4$

The ***DERIVATIVE*** of a function gives the ***GRADIENT*** (or rate of change) of that function at any point on the curve.

The derivative of a function $f(x)$ denoted by $f'(x)$ represents the:

- gradient of the function at a particular point
- gradient of the tangent line to the curve at a particular point
- instantaneous rate of change

2. FIRST PRINCIPLES & THE *RULES* OF DIFFERENTIATION

To determine the derivative $f'(x)$ of a function $f(x)$, two methods are used:

'Short method'
Differentiation RULES

- If $f(x) = x^n$, then $f'(x) = nx^{n-1}$

  (The rule in words: Multiply the coefficient of x by the exponent and then subtract 1 from the exponent.)

- Remember that if $f(x) = k$ (constant), then $f'(x) = 0$

- Before applying the rule make sure that you:
  1. Remove the brackets
  2. Remove surds
  3. Simplify by factorizing and dividing
  4. Remove all variables from the denominator.

'Long method'
First principles

Use the formula:
(on the Information sheet)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

N.B: Always remember to take out h as common factor in the numerator in order to cancel it out in the denominator before you can find the $\lim h \to 0$

DISTINGUISH BETWEEN

$f(3)$ is the ***y – coordinate*** at the point where $x = 3$

$f'(3)$ is the ***gradient (m)*** at the point where $x = 3$

NOTATION $\left[\frac{d}{dx} \quad \frac{dy}{dx} \quad f'(x) \quad D_x f(x) \right.$ ***all mean the same thing*** $\left. \right]$

Given: $f(x) = 3x^2$ *Then:* $f'(x) = 6x$	*Given:* $y = 3x^2$ *Then:* $\frac{dy}{dx} = 6x$

OR

Given: $D_x[3x^2]$ *Then:* $D_x[3x^2]$ $= 6x$	*Given:* $\frac{d}{dx}(3x^2)$ *Then:* $\frac{d}{dx}(3x^2)$ $= 6x$

3. EQUATIONS OF TANGENTS TO GRAPHS OF FUNCTIONS

- The average gradient on the curve between ***two points*** is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \textbf{or} \quad m = \frac{f(x+h) - f(x)}{h}$$

- The derivative gives the gradient of ***a point*** (one point) on a curve and that is also the gradient of a tangent to the curve at the given point.
- The derivative can be determined by using the first principle or the differentiation rule.
  Note: The first principles method is only used if instructed otherwise use the rule

- Method to determine the equation of a tangent to any graph:

  - Determine $f'(x)$ (the formula for the ***gradient*** of the tangent at any point on f)

  - Calculate the specific gradient at the *point of tangency* (i.e. the point where the tangent touches the graph) by substituting the x - value of the point into $f'(x)$
    (Now you have the m – value)

  - Substitute the x - value of the *point of tangency* into $f(x)$ to get the corresponding y - value of the *point of tangency*
    (Now you also have the point of tangency $(x\,;\,y)$)

  - Use $y = mx + c$ or $y - y_1 = m(x - x_1)$ to determine the equation of the tangent (straight line)

ACTIVITY 1

1.1 Determine the limit, if it exists:

1.1.1 $\lim_{x\to 2}(x+2)^2$ (2)

1.1.2 $\lim_{h\to 0}\frac{(3-h)^2-9}{h}$ (3)

1.2 Determine the derivative of $f(x) = 2x^2 - x$ from first principles. (5)

1.3 Determine $\frac{dy}{dx}$ if :

1.3.1 $y = (x + x^{-2})^2$ (4)

1.3.2 $xy^2 = 4$ (3)

1.4 If it is given that $f(x) = x^2 - \frac{4}{x^2}$,

1.4.1 Determine the gradient of the tangent at the point where $x = 2$. (3)

1.4.2 Determine the equation of the tangent in question 1.4.1 above. (3)

1.5 Given: $f(x) = -2x^2 + 1$

1.5.1 Show that the average gradient of the graph of f between the points where $x = 3$ and $x = 3 + h$, $(h \neq 0)$, is $-12 - 2h$. (4)

1.5.2 Use your answer in 1.5.1 to calculate $f'(3)$ from the first principles. (2)

1.5.3 Determine the numerical value of the gradient of the graph of f at $x = 0$. (1)

1.6 Differentiate y with respect to x. Leave your answer with positive exponents.

1.6.1 $y = (4 - 2x)^2$ (3)

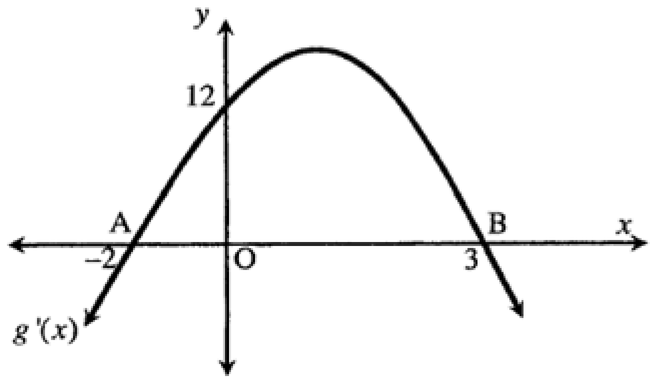
1.6.2 $y = \sqrt{x}(\sqrt{x} - a) + a$ (3)

ACTIVITY 2

2.1 Determine $f'(x)$ from first principles if $f(x) = 3x - x^2$. (5)

2.2 Determine $\frac{dy}{dx}$ if $y = \frac{3x}{5x^2} - \frac{1}{2\sqrt{x}}$. (4)

2.3 The graph of $y = g'(x)$ is sketched below, with x-intercepts at A(− 2; 0) and B(3; 0). The y-intercept of the sketched graph is (0; 12).



2.3.1 Determine the gradient of g at $x = 0$. (1)

2.3.2 For which value of ***x*** will the gradient of ***g*** be the same as the gradient in Question 2.3.1 (1)

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ACTIVITY 3

3.1 Given: $f(x) = 5 - 2x^2$

3.1.1 Determine $f'(x)$ from first principles: (5)

3.1.2 The line $g(x) = -\frac{1}{8}x + p$ is a tangent to the graph of f at the point A. Determine the coordinates of A. (4)

3.2 Determine:

3.2.1 $D_x\left[3x - 3x^2 - \frac{3}{x}\right]$ (4)

3.2.2 $\frac{dx}{dy}$, if $x = \frac{5}{4}.\sqrt[5]{y^2}$ (2)

3.3 Given: $f(x) = x^3 - 2x^2$
Determine the equation of the tangent to f at the point where $x = 2$. (6)

ACTIVITY 4

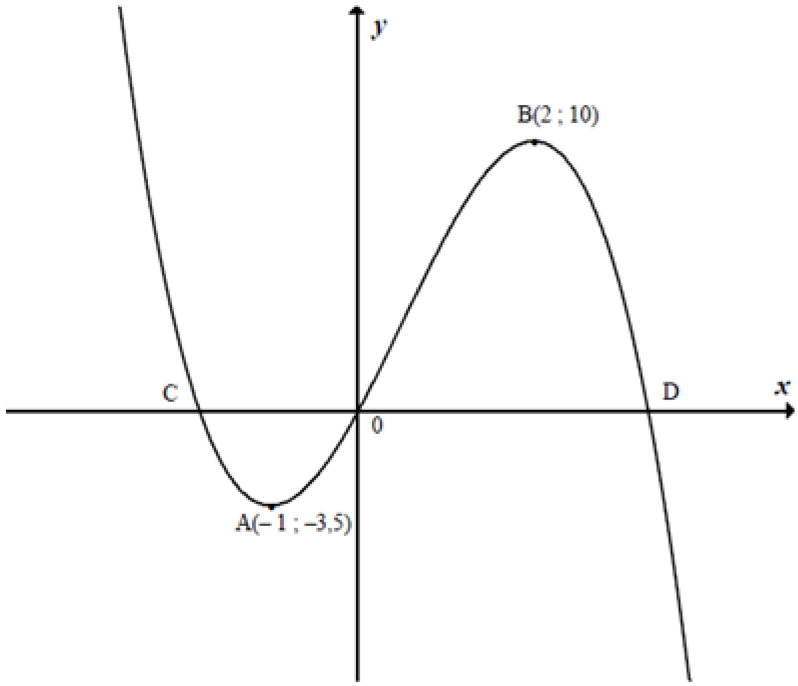
4.1 Differentiate from first Principles the function $f(x) = -\frac{3}{x}$ (5)

4.2 Determine y' using the rules of differentiation:

4.2.1 $y = \frac{x^2 - 3x + 1}{x^3}$ (4)

4.2.2 $y = \frac{\sqrt{t}}{2} - \frac{4}{7t^5}$ (4)

4.3 The graph of $h(x) = -x^3 + \frac{3}{2}x^2 + 6x$ is shown below. A(–1 ; 3,5) and B(2 ; 10) are the turning points of h. The graph passes through the origin and further cuts the x –axis at C and D.



4.3.1 Calculate the average gradient between A and B. (2)

4.3.2 Determine the equation of the tangent to h at $x = -2$. (5)

ACTIVITY 5

5.1 Given: $f(x) = -\frac{2}{x}$

5.1.1 Determine $f'(x)$ from first principles. (5)

5.1.2 For which value(s) of x will $f'(x) > 0$? Justify your answer. (2)

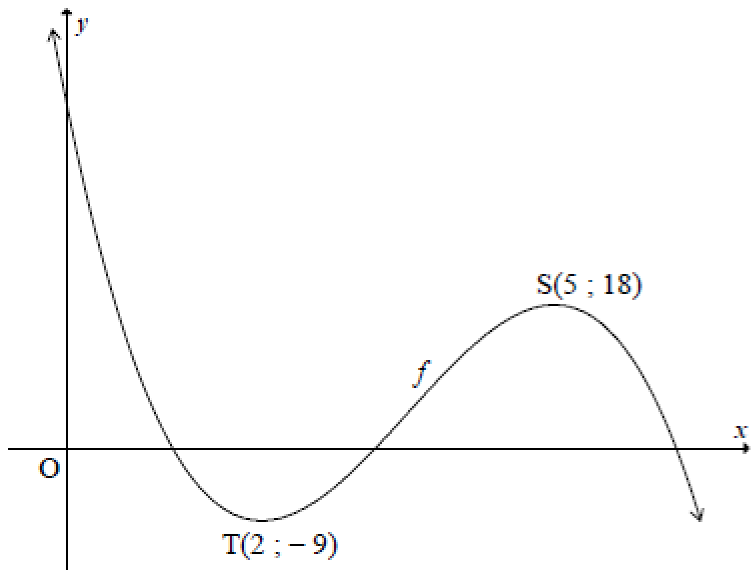
5.2 Determine $f'(x)$ from first principles if $f(x) = x^3$. (5)

5.3 If $y = (x^6 - 1)^2$, prove that $\frac{dy}{dx} = 12x^5\sqrt{y}$, if $x > 1$. (3)

5.4 Given: $y = 4(\sqrt[3]{x^2})$ and $x = w^{-3}$

Determine $\frac{dy}{dw}$. (4)

5.5 The function $f(x) = -2x^3 + 21x^2 - 60x + 43$ is sketched below.
The turning points of the graph of f are T(2 ; –9) and S(5 ; 18).



5.5.1 Determine an equation of the tangent to the graph of f at $x = 1$. (5)