UNIT IV

Amortization and Sinking Funds

7.1 AMORTIZATION OF A DEBT

The most common method of repaying an interest-bearing loan is by *amortization*. In this method, a series of periodic payments is made; when these payments are equal—the usual case — their size can be determined by the procedures of Chapters 5 and 6. (See Problem 7.1.)

The indebtedness at any time is called the *outstanding balance* or *outstanding principal*; it is just the discounted value of all unmade payments. Each sequential payment pays the interest on the unpaid balance and also repays a part of the outstanding principal. Over the term of the loan, as the outstanding principal decreases, the interest portion of each payment decreases and the principal portion increases. This shifting distribution is shown in an *amortization schedule* (Problems 7.4 and 7.5).

The common commercial practice is to round the payment up to the nearest cent; we shall follow this practice unless specified otherwise. However, instead of rounding up to the nearest cent, the lender may round up to the nearest dime or to the nearest dollar. Rounding up of the payment amount results in a reduced last payment, as may be determined from an equation of value at the time of the last payment. (See Problems 7.2 and 7.3.)

SOLVED PROBLEMS

7.1 A debt of \$6000 with interest at 16% compounded semiannually is to be amortized by equal semiannual payments of \$$R$ over the next 3 years, the first payment due in 6 months. Find the payment rounded up to the nearest cent.

The six payments of R form an ordinary simple annuity with $A = 6000$, $n = 6$, and $i = 0.08$. From (5.2),

$$R = \frac{A}{a_{\overline{n}|i}} = \frac{6000}{a_{\overline{6}|.08}} = \$1297.892317 \approx \$1297.90$$

7.2 Find the concluding payment in Problem 7.1.

Let X be the concluding payment. We arrange our data on a time diagram, Fig. 7-1, and set up an equation of value for X at time 6.

$$\begin{aligned} X + 1297.90 s_{\overline{5}|.08}(1.08) &= 6000(1.08)^6 \\ X + 8223.40 &= 9521.25 \\ X &= \$1297.85 \end{aligned}$$

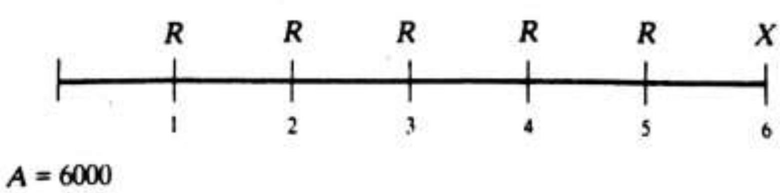

Fig. 7-1

7.3 Find the concluding payment X, if the payment in Problem 7.1 is rounded up to the nearest dollar.

Following Problem 7.2, with $R = \$1298$,

$$\begin{aligned} X + 1298 s_{\overline{5}|.08}(1.08) &= 6000(1.08)^6 \\ X + 8224.04 &= 9521.25 \\ X &= \$1297.21 \end{aligned}$$

7.4 Construct a complete amortization schedule for the debt of Problems 7.1 and 7.3.

Table 7-1

Payment Number	Periodic Payment	Interest at 8%	Principal Repaid	Outstanding Principal
				6000.00
1	1298.00	480.00	818.00	5182.00
2	1298.00	414.56	883.44	4298.56
3	1298.00	343.88	954.12	3344.44
4	1298.00	267.56	1030.44	2314.00
5	1298.00	185.12	1112.88	1201.12
6	1297.21	96.09	1201.12	0
TOTALS	7787.21	1787.21	6000.00	

See Table 7-1. Note that the interest due at the time of the first payment is 8% of \$6000, or \$480. The first payment of \$1298 will pay this interest and will also reduce the outstanding principal by $1298 - 480 = \$818$, bringing it to \$5182. The interest due at the time of the second payment is 8% of \$5182, or \$414.56; the procedure is repeated until the loan is repaid in full.

7.5 Mr. Adams borrows \$2000 to be repaid with quarterly payments over 2 years at $j_{12} = 24\%$. Construct a complete amortization schedule.

Find the rate i per quarter-year such that

$$\begin{aligned} (1+i)^4 &= (1.02)^{12} \\ i &= (1.02)^3 - 1 = 0.061208 \end{aligned}$$

The quarterly payment, R, is then

$$R = \frac{2000}{a_{\overline{8}|.061208}} = \$323.6134121 \approx \$323.62$$

Now we can construct the complete amortization schedule, Table 7-2.

Table 7-2

Payment Number	Periodic Payment	Interest at 6.1208%	Principal Repaid	Outstanding Principal
				2000.00
1	323.62	122.42	201.20	1798.80
2	323.62	110.10	213.52	1585.28
3	323.62	97.03	226.59	1358.69
4	323.62	83.16	240.46	1118.23
5	323.62	68.44	255.18	863.05
6	323.62	52.83	270.79	592.26
7	323.62	36.25	287.37	304.89
8	323.55	18.66	304.89	0
TOTALS	2588.89	588.89	2000.00	

In Tables 7-1 and 7-2, it should be noted that the total amount of principal repaid equals the original debt. The total of all periodic payments equals the total interest plus the total principal repaid. Finally, the entries in the Principal Repaid column (except for the final smaller payment) are in the ratio $1 + i$. For example, in Table 7-2,

$$\frac{213.52}{201.20} \cong \frac{226.59}{213.52} \cong \cdots \cong 1.061208$$

7.6 A loan is being repaid in equal installments at the end of each year for 10 years. Interest is at $j_1 = 10\%$. If the amount of principal repaid in the fifth payment is \$200, find (*a*) the amount of principal repaid in the eighth payment, (*b*) the amount of the loan (assume no rounding of the payments).

(*a*) Since the entries in the Principal Repaid column of the amortization schedule are in the ratio $1 + i$, the principal repaid in the eighth payment is $200(1.10)^3 = \$266.20$.

(*b*) The sum of the 10 repayments of principal is

$$200(1.1)^{-4} + \cdots + 200 + \cdots + 200(1.1)^5 = 200(1.1)^{-4} s_{\overline{10}|.1} = \$2177.10$$

and this must equal the amount of the loan.

7.7 Mrs. Smith borrows \$15 000 to be repaid in equal monthly installments over 4 years at $j_{12} = 9\%$. Find the total amount of interest she will pay in the lifetime of the loan.

The monthly payment, R, is

$$R = \frac{15\ 000}{a_{\overline{48}|.0075}} = 373.2756356$$

total value of all payments = $48 \times 373.2756356 = \$17\ 917.23$
total principal repaid = \$15 000
total interest paid = $17\ 917.23 - 15\ 000 = \$2917.23$

.8 Consider a loan of $\$A$ to be repaid with level payments of $\$R$ at the end of each period for n periods, at rate i per period. Show that in the kth line of the amortization schedule $(1 \le k \le n)$

(*a*) Outstanding principal $= R\, a_{\overline{n-k}|i}$

(*b*) Interest $= iR\, a_{\overline{n-k+1}|i} = R[1 - (1+i)^{-(n-k+1)}] = R[1 - v^{n-k+1}]$

(*c*) Principal repaid $= R(1+i)^{-(n-k+1)} = Rv^{n-k+1}$

(*a*) Outstanding principal after the $(k-1)$st payment is the discounted value of the remaining $n - (k-1) = n - k + 1$ payments, that is, $R\, a_{\overline{n-k+1}|i}$.

(*b*) Interest paid in the kth payment is

$$iR\, a_{\overline{n-k+1}|i} = iR\frac{1 - (1+i)^{-(n-k+1)}}{i} = R[1 - (1+i)^{-(n-k+1)}] = R[1 - v^{n-k+1}]$$

(*c*) Principal paid in the kth payment is

$$R - R[1 - (1+i)^{-(n-k+1)}] = R(1+i)^{-(n-k+1)} = Rv^{n-k+1}$$

7.9 A loan is being repaid with 20 annual installments at $j_1 = 9\%$. In what installment are the principal and interest portions most nearly equal to each other?

From Problem 7.8, in the kth payment:

$$\text{Interest} = R[1-(1+i)^{-(n-k+1)}] = R[1-(1.09)^{-(20-k+1)}]$$
$$\text{Principal} = R(1+i)^{-(n-k+1)} = R(1.09)^{-(20-k+1)}$$

We want to find k such that

$$\begin{aligned} R[1-(1.09)^{-(20-k+1)}] &= R(1.09)^{-(20-k+1)} \\ (1.09)^{-(20-k+1)} &= \tfrac{1}{2} \\ (1.09)^{20-k+1} &= 2 \\ (21-k)\log(1.09) &= \log 2 \\ -k\log 1.09 &= \log 2 - 21\log 1.09 \\ k &= \frac{21\log 1.09 - \log 2}{\log 1.09} \\ k &= 12.95676827 \end{aligned}$$

In the 13th payment, the principal and interest portions are most nearly equal.

7.2 OUTSTANDING PRINCIPAL

There are two methods that can be used to find the outstanding principal P on a debt A, being amortized by equal payments R over n periods at rate i per period (see Fig. 7-2).

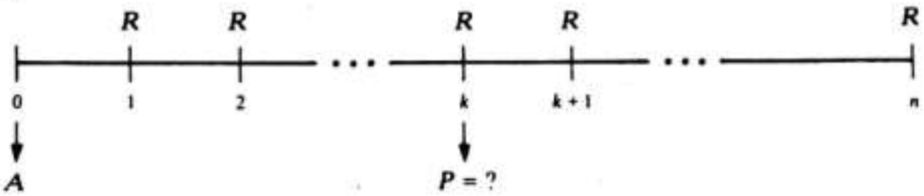

Fig. 7-2

Retrospective Method: Looking back in time, the outstanding principal P just after the kth payment is equal to the accumulated value of the debt less the accumulated value of the k payments made to date:

$$P = A(1+i)^k - R\, s_{\overline{k}|i} \tag{7.1}$$

Prospective Method: Looking ahead, the outstanding principal P just after the kth payment is equal to the discounted value of the $n-k$ payments yet to be made. If all payments, including the last one, are equal,

$$P = R\, a_{\overline{n-k}|i} \tag{7.2}$$

While (7.1) and (7.2) are algebraically equivalent (see Problem 7.8), (7.2) can only be used if all remaining payments are equal. We have seen that most loans have a concluding irregular (usually smaller) payment. In such situations, the Prospective Method is still applicable provided (7.2) is suitably modified. (See Problem 7.11.)

The amortization method is quite often used to pay off loans incurred in purchasing a property. In such cases, the outstanding principal is called the *seller's equity*. The amount of principal that has been paid already, plus the down payment, is called the *buyer's equity* or the *owner's equity*. Clearly,

buyer's equity + seller's equity = selling price

(See Problem 7.14.) This formula does not account for any change in the value of the property.

SOLVED PROBLEMS

7.10 Show that (7.1) and (7.2) are equivalent if all payments are equal.

Substituting $A = R\, a_{\overline{n}|i}$ in (7.1),

$$\begin{aligned} P &= R\, a_{\overline{n}|i}(1+i)^k - R\, s_{\overline{k}|i} = R\left[\frac{1-(1+i)^{-n}}{i}\right](1+i)^k - R\left[\frac{(1+i)^k - 1}{i}\right] \\ &= R\frac{(1+i)^k - (1+i)^{-(n-k)} - (1+i)^k + 1}{i} = R\frac{1-(1+i)^{-(n-k)}}{i} \\ &= R\, a_{\overline{n-k}|i} \end{aligned}$$

7.11 The amortization schedule in Problem 7.4 shows an outstanding balance of \$2314.00 after four payments (2 years). Confirm this balance, using (*a*) the retrospective, (*b*) the prospective, method.

(*a*) With $A = 6000$, $R = 1298$, $k = 4$, and $i = 0.08$, (7.1) gives

$$P = 6000(1.08)^4 - 1298 s_{\overline{4}|.08} = 8162.93 - 5843.93 = \$2314.00$$

(*b*) With $R = 1298$, $k = 4$, $n = 6$, and $i = 0.08$, (7.2) gives

$$P = 1298 a_{\overline{2}|.08} = \$2314.68$$

This is an incorrect answer. From Problem 7.3, the concluding payment is \$1297.21; hence, the discounted value of the remaining two payments is actually

$$P = 1298(1.08)^{-1} + 1297.21(1.08)^{-2} = 1201.85 + 1112.15 = \$2314.00$$

7.12 A loan of \$8000 is to be amortized with equal monthly payments over 2 years at $j_{12} = 15\%$. Find the outstanding principal after 7 months and split the eighth payment into principal and interest portions.

The monthly payment is

$$R = \frac{8000}{a_{\overline{24}|.0125}} = \$387.8931844 \approx \$387.90$$

The outstanding principal P after 7 payments is

$$P = 8000(1.0125)^7 - 387.90 s_{\overline{7}|.0125} = \$5907.53$$

The interest portion of the eighth payment is

$$(5907.53)(0.0125) = \$73.84$$

And the principal portion is $387.90 - 73.84 = \$314.06$.

7.13 Solve Problem 7.12 assuming $j_\infty = 16\%$.

Find the rate i per month such that

$$\begin{aligned}(1+i)^{12} &= e^{0.16}\\ i &= 0.013422619\end{aligned}$$

The monthly payment R is then

$$R = \frac{8000}{a_{\overline{24}|.013422619}} = \$392.1145563 \approx \$392.12$$

$$P(\text{after 7 months}) = 8000(1.013422619)^7 - 392.12s_{\overline{7}|.013422619} = \$5924.75$$

The interest portion of the eighth payment is

$$(5924.75)(0.013422619) = \$79.53$$

and the principal portion is $392.12 - 79.53 = \$312.59$.

7.14 Harry buys a cottage worth \$42 000 by paying \$7000 down and the balance with interest at $j_{12} = 9\%$ in monthly installments of \$600 for as long as necessary. Find Harry's equity at the end of 5 years.

Using the retrospective method, we calculate the seller's equity at the end of 5 years:

$$\begin{aligned}P &= 35\,000(1.0075)^{60} - 600s_{\overline{60}|.0075}\\ &= 54\,798.84 - 45\,254.48 = \$9544.36\end{aligned}$$

Then, buyer's equity $= 42\,000 - 9544.36 = \$32\,455.64$

7.15 The Andersons borrow \$15 000 to buy a car. The loan will be repaid over three years with monthly payments at $j_{12} = 6\%$. Find the total interest paid in the 12 payments of the second year.

The monthly payment is

$$R = \frac{15\,000}{a_{\overline{36}|.005}} = \$456.3290618 \approx \$456.33$$

Using (7.1), the outstanding principal after 1 year is

$$P = 15\,000(1.005)^{12} - 456.33s_{\overline{12}|.005} = 15\,925.17 - 5629.09 = \$10\,296.08$$

and the outstanding principal after 2 years is

$$P = 15\,000(1.005)^{24} - 456.33s_{\overline{24}|.005} = 16\,907.40 - 11\,605.36 = \$5302.04$$

The total principal repaid in the second year is therefore

$$10\,296.08 - 5302.04 = \$4994.04$$

The balance of the 12 payments made in the second year represents interest. Hence,

$$\text{interest paid} = (12)(456.33) - 4994.04 = \$481.92$$

7.3 MORTGAGES

Buying a house is likely to be the most expensive purchase an individual or couple ever makes. Usually, a large loan, or *mortgage*, is taken out when one buys a home, which is ordinarily repaid in monthly installments over a long period (25 to 30 years). The lender holds the legal right to repossess the property upon failure to repay.

Prevailing practice in the U.S. is to have monthly payments on mortgage loans, with interest compounded monthly. Because of today's unpredictable rates of interest, traditional fixed-rate mortgages have been replaced by a variety of creative financing arrangements—it is important to understand the costs of these complex contracts.

Canadian mortgage regulations require that the interest can be compounded, at most, semiannually, whereas mortgage payments are usually made monthly. Thus mortgage amortizations in Canada are, in effect, general annuities. (See Problem 7.21.)

SOLVED PROBLEMS

7.16 A couple buys a condominium on May 1, 1994, for \$65 000. They make a 20% down payment and get a 29-year mortgage loan at $j_{12} = 10\%$ for the balance; the loan is to be amortized by equal monthly payments rounded up to the nearest dime. If they make the first payment on June 1, 1994, how much interest can they deduct when they prepare their income tax return for 1994? Show the first 3 lines and last 3 lines of the amortization schedule.

As the down payment is 20% of \$65 000, or \$13 000, the couple borrows \$52 000. Thus, $A = 52\ 000$, $n = 29 \times 12 = 348$, $i = \frac{5}{6}\%$; and we calculate the monthly payment as

$$R = \frac{52\ 000}{a_{\overline{348}|5/600}} = \$458.90 \text{ (rounded up to the nearest dime)}$$

Figure 7-3 shows the payments made during 1994. Outstanding principal on December 1, 1994, is

$$P_7 = 52\ 000\left(\frac{605}{600}\right)^7 - 458.90 s_{\overline{7}|5/600} = 55\ 110.23 - 3293.73 = \$51\ 816.50$$

so that total principal repaid during 1994 is $52\ 000 - 51\ 816.50 = \$183.50$. Consequently, total interest paid during 1994 is

$$(7)(458.90) - 183.50 = \$3028.80$$

They can list \$3028.80 on their income tax return as a mortgage interest deduction.

	458.90	458.90	458.90	458.90	458.90	458.90	458.90
0	1	2	3	4	5	6	7
May 1	June 1	July 1	Aug. 1	Sept. 1	Oct. 1	Nov. 1	Dec. 1
$A = 52\ 000$							

Fig. 7-3

The beginning and the end of the amortization schedule are given in Table 7-3. To compute the last 3 lines, we first find the outstanding principal after the 345th payment:

$$P_{345} = 52\ 000\left(\frac{605}{600}\right)^{345} - 458.90 s_{\overline{345}|5/600} = 910\ 806.68 - 909\ 476.27 = \$1330.41$$

The last monthly payment is smaller, owing to the rounding of the regular monthly payment. Table 7-3 shows that at the beginning the payments go mainly toward interest due, and at the end mainly toward principal.

Table 7-3

Payment Number	Monthly Payment	Interest at $\frac{5}{6}\%$	Principal Repaid	Outstanding Principal
				52 000.00
1	458.90	433.33	25.57	51 974.43
2	458.90	433.12	25.78	51 948.65
3	458.90	432.91	25.99	51 922.66
....	....	....	....	....
345				1 330.41
346	458.90	11.09	447.81	882.60
347	458.90	7.36	451.54	431.06
348	434.65	3.59	431.06	0

7.17 In an effort to advertise low rates of interest, but still achieve high rates of return, lenders sometimes charge *points*. Each point is a 1% discount from the face value of the loan. Suppose that a cottage is being sold for \$55 000 and that the buyer pays \$15 000 down and gets a \$40 000, 15-year mortgage at $j_{12} = 9\%$ from a lender who charges 5 points. What is the true interest rate on the loan?

The lender will advance 95% of \$40 000, or \$38 000. However, the monthly payment is calculated on the basis of a \$40 000 loan:

$$R = \frac{40\ 000}{a_{\overline{180}|.0075}} = \$405.71$$

Thus, the true monthly interest rate i is the solution of the equation

$$405.71 a_{\overline{180}|i} = 38\ 000 \qquad \text{or} \qquad a_{\overline{180}|i} = 93.6630$$

Applying the method of interpolation (Section 5.6), we have:

$$5.5360\left\{\begin{array}{l} 4.9304\left\{\begin{array}{c|c} a_{\overline{180}|i} & j_{12} \\ \hline 98.5934 & 9\% \\ 93.6630 & j_{12} \end{array}\right\} x \\ \qquad\quad \begin{array}{c|c} 93.0574 & 10\% \end{array} \end{array}\right\} 1\% \qquad\qquad \frac{x}{1\%} = \frac{4.9304}{5.5360} \quad x = 0.89\% \quad j_{12} = 9.89\%$$

The true interest rate on the mortgage loan is $j_{12} = 9.89\%$.

Points are charged at the beginning of the mortgage loan. Borrowers who decide to pay off their mortgage loan ahead of time will not recover any part of this charge. As a result, they will actually pay an even higher interest rate, as illustrated in Problem 7.18.

7.18 The borrower in Problem 7.17 sells his cottage at the end of 5 years and pays off the mortgage. What rate did the borrower pay?

The outstanding principal at the end of 5 years, *just before* the 60th payment, is

$$\begin{aligned} P &= 40\,000(1.0075)^{60} - 405.71 s_{\overline{59}|.0075}(1.0075) \\ &= 62\,627.24 - 30\,194.62 = \$32\,432.62 \end{aligned}$$

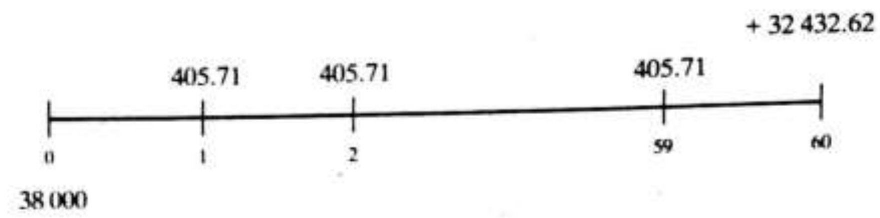

Fig. 7-4

We arrange our data as shown in Fig. 7-4. We want to find the rate j_{12} which makes the discounted value, A', of the payments, including the last payment of \$32 432.62, equal to \$38 000. Choosing the trial values $j_{12} = 10\%$ and $j_{12} = 11\%$, we calculate:

$$A'_{10\%} = 405.71 a_{\overline{59}|5/600} + 32\,432.62\left(\frac{605}{600}\right)^{-60} = 18\,848.31 + 19\,712.18 = \$38\,560.49$$

$$A'_{11\%} = 405.71 a_{\overline{59}|11/1200} + 32\,432.62\left(\frac{1211}{1200}\right)^{-60} = 18\,425.17 + 18\,758.94 = \$37\,184.11$$

Then, using linear interpolation, we get

		A'	j_{12}		
1376.38	560.49	38 560.49	10%	x	1%
		38 000.00	j_{12}		
		37 184.11	11%		

$$\frac{x}{1\%} = \frac{560.49}{1376.38}$$

$$x = 0.41\%$$

$$j_{12} = 10.41\%$$

The true rate is $j_{12} = 10.41\%$ if the borrower pays off the mortgage at the end of 5 years.

7.19 With mortgage rates at $j_{12} = 12\%$, the ABC Savings and Loans Company makes a special offer to its customers. It will lend mortgage money and determine the monthly payment for the next 5 years as if $j_{12} = 9\%$. Over the 5-year period, the mortgage will be carried at $j_{12} = 12\%$ and any deficiency that results will be added to the outstanding balance to be refinanced in 5 years' time. If the Browns are taking out a \$90 000 mortgage to be repaid over 25 years under this scheme, find their outstanding balance at the end of 5 years.

The monthly payment R, assuming $j_{12} = 9\%$, is

$$R = \frac{90\,000}{a_{\overline{300}|.0075}} = \$755.2767273 \approx \$755.28$$

That is, they owe \$11 819.23 more than they originally borrowed. This balance will be refinanced at the new prevailing rate of interest.

7.20 XYZ Savings and Loans issues mortgages where payments are determined by the rate of interest that prevails on the day the loan is made. After that, the rate of interest varies according to market forces but the monthly payments do not change in dollar size. Instead, the length of time to full repayment is either lengthened (if interest rates rise) or shortened (if interest rates fall). Mr. Adams takes out a 20-year, \$70 000 mortgage at $j_{12} = 9\%$. After exactly 2 years (24 payments) interest rates change. Find the duration of the loan and the final smaller payment if the new interest rate stays fixed at (*a*) $j_{12} = 10\%$, (*b*) $j_{12} = 8\%$.

The monthly amount R to repay a \$70 000 loan over 20 years at $j_{12} = 9\%$ ($i = 0.0075$) is

$$R = \frac{70\,000}{a_{\overline{240}|\,.0075}} = \$629.8081691 \approx \$629.81$$

The outstanding principal after 2 years is, by (7.1),

$$\begin{aligned} P &= 70\,000(1.0075)^{24} - 629.81 s_{\overline{24}|\,.0075} \\ &= 83\,748.95 - 16\,493.76 = \$67\,255.19 \end{aligned}$$

Payments remains at \$629.81, but the duration of the loan will change.

(*a*) If $j_{12} = 10\%$ ($i = .1/12$), we find n such that

$$\begin{aligned} 67\,255.19 &= 629.81 a_{\overline{n}|i} \\ a_{\overline{n}|i} &= 106.7864753 \\ \frac{1-(1+i)^{-n}}{i} &= 106.7864753 \\ (1+i)^{-n} &= 0.110112706 \\ n &= -\frac{\log 0.110112706}{\log(1+i)} = 265.8517003 \end{aligned}$$

Thus there are 265 more payments of \$629.81, plus a final smaller payment of

$$\begin{aligned} X &= 67\,255.19(1+i)^{266} - 629.81 s_{\overline{265}|i}(1+i) \\ &= 611\,537.17 - 611\,000.43 = \$536.74 \end{aligned}$$

The total duration of the loan is $24 + 265 + 1 = 290$ months, or 24 years 2 months.

(*b*) If $j_{12} = 8\%$ ($i = .08/12$), we find n such that

$$\begin{aligned} 67\,255.19 &= 629.81 a_{\overline{n}|i} \\ a_{\overline{n}|i} &= 106.7864753 \\ (1+i)^{-n} &= 0.288090165 \\ n &= -\frac{\log 0.288090165}{\log(1+i)} = 187.2938181 \end{aligned}$$

Thus there are 187 more payments of \$629.81, plus a final smaller payment of

$$\begin{aligned} X &= 67\,255.19(1+i)^{188} - 629.81 s_{\overline{187}|i}(1+i) \\ &= 234\,549.87 - 234\,364.38 = \$185.49 \end{aligned}$$

The total duration of this loan is $24 + 187 + 1 = 212$ months, or 17 years 8 months.

7.21 An advertisement by the Royal Trust in Canada said:

"Our new Double-Up mortgage can be paid off faster and that dramatically reduces the interest you pay. You can double your payment any or every month, with no penalty. Then your payment reverts automatically to its normal amount the next month.

What this means to you is simple. You will pay your mortgage off sooner. And that's good news because you can save thousands of dollars in interest as a result."

Consider a \$120 000 mortgage at 8% per annum compounded semiannually, amortized over 25 years with no anniversary prepayments.

(*a*) Find the required monthly payment for the above mortgage.

(*b*) What is the total amount of interest paid over the full amortization period (assuming $j_2 = 8\%$)?

(*c*) Suppose that payments were "Doubled-Up," according to the advertisement, every sixth and twelfth month.

(i) How many years and months would be required to pay off the mortgage?

(ii) What would be the total amount of interest paid over the full amortization period?

(iii) How much of the loan would still be outstanding at the end of 3 years, just after the Doubled-Up payment then due?

(iv) How much principal is repaid in the payment due in 37 months?

(*a*) First we find the rate i per month such that

$$\begin{aligned}(1+i)^{12} &= (1.04)^2 \\ i &= (1.04)^{1/6} - 1 = 0.006558197\end{aligned}$$

The monthly payment R is

$$R = \frac{120\ 000}{a_{\overline{300}|i}} = 915.8561463 \approx \$915.86$$

(*b*) The concluding payment $= 120\ 000(1+i)^{300} - 915.86 s_{\overline{299}|i}(1+i)$
$= 852\ 802.00 - 851\ 889.73 = \$912.27$

$$\begin{aligned}\text{The total amount of interest} &= (299 \times 915.86 + 912.27) - 120\ 000 \\ &= \$154\ 754.41\end{aligned}$$

(*c*) (i) We calculate an equivalent semiannual payment

$$915.86 s_{\overline{6}|i} + 915.86 = \$6501.91$$

We find n such that

$$\begin{aligned}6501.91 a_{\overline{n}|.04} &= 120\ 000 \\ \frac{1-(1.04)^{-n}}{.04} &= \tfrac{120\ 000}{6501.91} \\ (1.04)^{-n} &= 0.261755392 \\ n &= -\frac{\log 0.261755392}{\log 1.04} = 34.17441251\end{aligned}$$

It would require 34.17441251 half-years, that is, 17 years and 2 months to pay off the mortgage.

(ii) The concluding payment at the end of 17 years and 2 months (206th payment) is

$$\begin{aligned} &120\,000(1+i)^{206} - 6501.91s_{\overline{34}|.04}(1+i)^2 - 915.86(1+i) \\ &= 461\,309.67 - 460\,186.96 - 921.87 = \$200.84 \end{aligned}$$

$$\begin{aligned} \text{The total amount of interest} &= [(205+34)\times 915.86 + 200.84] - 120\,000 \\ &= \$99\,091.38 \end{aligned}$$

(iii) The outstanding principal at the end of 3 years is

$$120\,000(1.04)^6 - 6501.91s_{\overline{6}|.04} = \$108\,711.27$$

(iv) The interest paid in the 37th payment $= 108\,711.27i = \$712.95$
The principal paid in the 37th payment $= 915.86 - 712.95 = \$202.91$

7.4 REFINANCING A LOAN

Amortization Method

Often borrowers will want to renegotiate long-term loans, especially if interest rates have dropped. Most contracts stipulate penalties at the time of any early renegotiation.

If refinancing a loan at a lower rate is being contemplated, the savings due to the lower interest charges must be compared with the cost of refinancing to decide whether the refinancing would be profitable.

Sum-of-Digits Method

This method, which has a built-in prepayment penalty, was initially used as a close approximation to the amortization method. (See Problem 7.27.) At high rates of interest, however, the outstanding principal as determined by the sum-of-digits method leads to a significant penalty against the borrower. (See Problem 7.31.)

SOLVED PROBLEMS

7.22 A borrower has an \$8000 loan with the Easy-Credit Finance Company. The loan is to be repaid over 4 years at $j_{12} = 18\%$. The contract stipulates a penalty in case of early repayment, equal to 3 months' payments. Just after the 20th payment, the borrower determines that his local bank would lend him the money at $j_{12} = 13.5\%$. Should he refinance?

The original monthly payment, R_1, at $j_{12} = 18\%$ is

$$R_1 = \frac{8000}{a_{\overline{48}|.015}} = \$234.9999969 \approx \$235$$

The outstanding balance P after 20 payments is

$$P = 8000(1.015)^{20} - 235s_{\overline{20}|.015} = 10\,774.84 - 5434.06 = \$5340.78$$

Adding the penalty, we obtain as the total sum to be refinanced $5340.78 + 3(235) = \$6045.78$

The new monthly payment, R_2, at $j_{12} = 13\frac{1}{2}\%$ would be

$$R_2 = \frac{6045.78}{a_{\overline{28}|.01125}} = 252.9130458 \approx \$252.92$$

Since R_2 exceeds the original monthly payment of \$235, the borrower should not refinance.

7.23 A couple purchased a home and signed a mortgage contract for \$105 000 to be paid in monthly installments over 25 years at $j_{12} = 10\frac{1}{2}\%$. The contract stipulates that after 5 years the mortgage will be renegotiated at the new prevailing rate of interest. Find (*a*) the monthly payment for the initial 5-year period, (*b*) the outstanding principal after 5 years, (*c*) the new monthly payment after 5 years at $j_{12} = 9\%$.

(*a*) With $A = 105\ 000$, $n = 300$, and $j_{12} = 10\frac{1}{2}\%$ $(i = 0.00875)$, we calculate the monthly payment for the initial 5-year period:

$$R_1 = \frac{105\ 000}{a_{\overline{300}|.00875}} = \$991.3907904 \approx \$991.40$$

(*b*)

$$\begin{aligned} P &= 105\ 000(1.00875)^{60} - 991.40 s_{\overline{60}|.00875} \\ &= 177\ 093.31 - 77\ 794.08 = \$99\ 299.23 \end{aligned}$$

Note that in 5 years, with payments totaling $(60)(991.40) = \$59\ 484$, only

$$105\ 000 - 99\ 299.23 = \$5700.77$$

of principal has been repaid.

(*c*) Using $A = 99\ 299.23$, $n = 240$, $j_{12} = 9\%$ $(i = 0.0075)$, we calculate the new monthly payment after 5 years:

$$R_2 = \frac{99\ 299.23}{a_{\overline{240}|.0075}} = \$893.4209463 \approx \$893.43$$

Hence, their monthly payment will fall by \$97.97.

7.24 Solve Problem 7.23(*c*) assuming $j_{12} = 12\%$.

Now $i = 0.01$, and

$$R_2 = \frac{99\ 299.23}{a_{\overline{240}|.01}} = \$1093.370052 \approx \$1093.38$$

Hence, their monthly payment would rise by \$101.98.

7.25 Elizabeth is repaying a debt of \$5000 with monthly payments over 3 years at $j_{12} = 16\frac{1}{2}\%$. At the end of the first year she makes an extra single payment of \$500. She then shortens the repayment period by 1 year and renegotiates the loan without an interest rate change. Find the new monthly payment and the amount of interest she saves by the refinancing.

We have $A = 5000$, $n = 36$, $j_{12} = 16\frac{1}{2}\%$ $(i = 0.01375)$; and so the original monthly payment is

$$R_1 = \frac{5000}{a_{\overline{36}|.01375}} = \$177.0219127 \approx \$177.03$$

and the total interest in the original loan is $(36)(177.0219127) - 5000 = \$1372.79$.

The outstanding principal after 1 year is

$$5000(1.01375)^{12} - 177.03 s_{\overline{12}|.01375} = 5890.34 - 2292.61 = \$3597.73$$

After the extra payment, we have $A = 3597.73 - 500 = \$3097.73$, $n = 12$, $j_{12} = 16\frac{1}{2}\%$ $(i = 0.01375)$; the new monthly payment is

$$R_2 = \frac{3097.73}{a_{\overline{12}|.01375}} = \$281.7931766 \approx \$281.80$$

and the total interest in the revised loan is

$$(12)(177.03) + 500 + (12)(281.7931766) - 5000 = \$1005.88$$

Therefore, her savings in interest is $1372.79 - 1005.88 = \$366.91$.

7.26 A consumer borrows \$10 000 to be repaid in monthly installments over 1 year at $j_{12} = 12\%$. Construct a complete amortization schedule.

Having calculated

$$R = \frac{10\ 000}{a_{\overline{12}|.01}} = \$888.4878868 \approx \$888.49$$

we construct Table 7-4.

Table 7-4

Payment Number	Periodic Payment	Interest at 1%	Principal Repaid	Outstanding Principal
				\$10 000.00
1	888.49	100.00	788.49	9 211.51
2	888.49	92.12	796.37	8 415.14
3	888.49	84.15	804.34	7 610.80
4	888.49	76.11	812.38	6 798.42
5	888.49	67.98	820.51	5 977.91
6	888.49	59.78	828.71	5 149.20
7	888.49	51.49	837.00	4 312.20
8	888.49	43.12	845.37	3 466.83
9	888.49	34.67	853.82	2 613.01
10	888.49	26.13	862.36	1 750.65
11	888.49	17.51	870.98	879.67
12	888.47	8.80	879.67	0
TOTALS	10 661.86	661.86	10 000.00	

7.27 For Problem 7.26, construct a complete repayment schedule using the sum-of-digits method.

By (2.2), the sum of digits from 1 to 12 is

$$\frac{12}{2}(1 + 12) = 78$$

Under the sum-of-digits method, the total interest in the loan, \$661.86, is allocated 12/78 to the first payment, 11/78 to the second payment, ..., 1/78 to the twelfth payment. This gives the schedule of Table 7-5.

Table 7-5

Payment Number	Periodic Payment	Interest Allocated	Principal Repaid	Outstanding Principal
				\$10 000.00
1	888.49	101.82	786.67	9 213.33
2	888.49	93.34	795.15	8 418.18
3	888.49	84.85	803.64	7 614.54
4	888.49	76.37	812.12	6 802.42
5	888.49	67.88	820.61	5 981.81
6	888.49	59.40	829.09	5 152.72
7	888.49	50.91	837.58	4 315.14
8	888.49	42.43	846.06	3 469.08
9	888.49	33.94	854.55	2 614.53
10	888.49	25.46	863.03	1 751.50
11	888.49	16.97	871.52	879.98
12	888.47	8.49	879.98	0
TOTALS	10 661.86	661.86	10 000.00	

A comparison of Tables 7-4 and 7-5 leads to the following conclusions:

(i) Each outstanding principal balance under the sum-of-digits method exceeds the true balance under the amortization method. Hence, if the loan is refinanced before the end of its term, there is a built-in penalty for the borrower.

(ii) If the loan is paid off full-term, the total amount of interest is the same as under the amortization method, and no penalty is incurred.

7.28 For Problem 7.27, calculate the outstanding principal after five payments, without constructing the sum-of-digits schedule.

A banking institution would normally use the following format:

$$\begin{aligned}
&\text{Total Original Debt} = 12 \times 888.4878868 && = \$10\,661.85\\
&\text{Less: Interest Not Yet Due} = \left(\frac{1+2+3+\cdots+7}{78}\right)(661.85) && \\
&\qquad = \left(\tfrac{28}{78}\right)(661.85) && = 237.59\\
&\text{Less: Payments Already Made} = 5 \times 888.49 && = 4\,442.45\\
&\text{Outstanding Principal} && = \$5\,981.81
\end{aligned}$$

7.29 A \$15 000 loan is to be repaid over 10 years at $j_{12} = 18\%$. Construct the first three lines of the repayment schedule, using (a) the amortization method, (b) the sum-of-digits method.

$$R = \frac{15\,000}{a_{\overline{120}|.015}} = \$270.2777986 \approx \$270.28$$

(a) The beginning of the amortization schedule is given in Table 7-6.

Table 7-6

Payment Number	Periodic Payment	Interest at $1\frac{1}{2}\%$	Principal Repaid	Outstanding Principal
				\$15 000.00
1	270.28	225.00	45.28	14 954.72
2	270.28	224.32	45.96	14 908.76
3	270.28	223.63	46.65	14 862.11

(*b*) The total interest is $(120)(270.2777986) - 15\ 000 = \$17\ 433.34$, and the sum of digits from 1 to 120 is

$$\frac{120}{2}(1 + 120) = 7260$$

Thus, the interest portion of the first payment is

$$\left(\frac{120}{7260}\right)(17\ 433.34) = \$288.15$$

and so on. This first apportionment of interest exceeds the size of the first payment, leading to a repayment schedule that begins as in Table 7-7.

Table 7-7

Payment Number	Periodic Payment	Interest Allocated	Principal Repaid	Outstanding Principal
				\$15 000.00
1	270.28	288.15	−17.87	15 017.87
2	270.28	285.75	−15.47	15 033.34
3	270.28	283.35	−13.07	15 046.41

7.30 For the loan of Problem 7.29, find the outstanding principal after 2 years (24 payments), using (*a*) the amortization method, (*b*) the sum-of-digits method.

(*a*) $P = 15\ 000(1.015)^{24} - 270.28 s_{\overline{24}|.015} = 21\ 442.54 - 7739.07 = \$13\ 703.47$

(*b*)

Total Debt $= 120 \times 270.2777986$	=	\$32 433.34
Less: Interest Not Yet Due $= (\frac{4656}{7260})(17\ 433.34)$	=	11 180.39
Less: Payments to Date $= 24 \times 270.28$	=	6 486.72
Outstanding Principal	=	\$14 766.23

7.31 For the loan in Problem 7.29, the borrower determines after 2 years of payments that he can renegotiate the loan at $j_{12} = 15\%$ over the remaining 8 years. Should he renegotiate the loan, if the lending institution uses the sum-of-digits method to determine the outstanding principal?

If he renegotiates, he will have to pay off the outstanding principal of \$14 766.23 [Problem 7.30 (*b*)] over 8 years, with new monthly payments R at $j_{12} = 15\%$.

$$R = \frac{14\ 766.23}{a_{\overline{96}|.0125}} = \$264.9859823 \approx \$264.99$$

Since the new monthly payment is less than \$270.28, he should renegotiate.

7.32 In Problem 7.31, what would the new monthly payment have been under the amortization method at $j_{12} = 15\%$?

Using the outstanding principal calculated in Problem 7.30(*a*), the new monthly payment is

$$R = \frac{13\ 703.47}{a_{\overline{96}|.0125}} = \$245.9143233 \approx \$245.92$$

7.5 SINKING FUNDS

A specified sum of money can be accumulated by a specified future date through periodic deposits into a *sinking fund.* A schedule showing how a sinking fund accumulates to the desired amount is called a *sinking-fund schedule.*

The Sinking-Fund Method of Retiring a Debt

A common method of paying off a long-term loan is for the borrower to pay the interest on the loan as it falls due and to create a sinking fund to accumulate the principal at the end of the term of the loan. The sum of the interest payment and the sinking-fund deposit is called the *periodic expense* or *periodic cost of the debt.* The *book value* of the debt at any time is the original principal less the amount in the sinking fund at that time.

SOLVED PROBLEMS

7.33 A company wants to save \$100 000 over the next 5 years so that they can expand their plant facility. How much must be deposited at the end of each year if their money earns interest at $j_1 = 6\%$? Construct a complete sinking-fund schedule.

The sinking-fund deposits form an ordinary simple annuity, with $S = 100\ 000$, $i = 0.06$, and $n = 5$. From (5.1),

$$R = \frac{100\ 000}{s_{\overline{5}|.06}} = \$17\ 739.64$$

which leads to the sinking-fund schedule of Table 7-8. There is a 1¢ roundoff error in the final amount.

Table 7-8

Deposit Number	Deposit	Interest on Fund at 6%	Increase in Fund	Amount in Fund at End of Period
1	17 739.64	0	17 739.64	17 739.64
2	17 739.64	1064.38	18 804.02	36 543.66
3	17 739.64	2192.62	19 932.26	56 475.92
4	17 739.64	3388.56	21 128.20	77 604.12
5	17 739.64	4656.25	22 395.89	100 000.01

7.34 The Smiths want to save \$12 000 for a down payment on a house. If they save \$500 a month in an account paying interest at $j_{12} = 4.5\%$, how many deposits will be required and what will be the size of the final smaller deposit?

Find n such that $500s_{\overline{n}|.00375} = 12\ 000$ or $s_{\overline{n}|.00375} = 24$, whence

$$\frac{(1.00375)^n - 1}{.00375} = 24$$
$$(1.00375)^n = 1.09$$
$$n = \frac{\ln 1.09}{\ln 1.00375} = 23.02378096$$

There will be 23 deposits of \$500, plus a final deposit of

$$X = 12\ 000 - 500s_{\overline{23}|.00375}(1.00375) = 12\ 000 - 12\ 032.02 = -\$32.02$$

The negative sign on X tells us (as in the second part of Problem 5.35) that it takes 24 months to save the \$12 000, but that no 24th deposit is required.

7.35 A city needs to have \$200 000 at the end of 15 years to retire a bond issue. What annual deposits are necessary if their money earns interest at $j_\infty = 12\frac{1}{2}\%$?

Find the rate i per year such that

$$1 + i = e^{0.125} \quad \text{or} \quad i = 0.133148453$$

With $S = 200\ 000$, $n = 15$, $i = 0.133148453$, (5.1) gives

$$R = \frac{200\ 000}{s_{\overline{15}|.133148453}} = \$4823.50$$

7.36 Construct the first three and last three lines of the sinking-fund schedule for Problem 7.35.

In order to complete the last three lines of the schedule, without doing the full schedule, we determine the amount in the sinking fund at the end of 12 years:

$$4823.50 s_{\overline{12}|i} = \$126\ 129.35$$

and complete the schedule from that point. See Table 7-9. There is a 13¢ roundoff error in the final amount.

Table 7-9

Deposit Number	Deposit	Interest on Fund at i	Increase in Fund	Amount in Fund at End of Period
1	4823.50	0	4 823.50	4 823.50
2	4823.50	642.24	5 465.74	10 289.24
3	4823.50	1 370.00	6 193.50	16 482.74
....	....	....	....	....
12				126 129.35
13	4823.50	16 793.93	21 617.43	147 746.78
14	4823.50	19 672.26	24 495.76	172 242.54
15	4823.50	22 933.83	27 757.33	199 999.87

7.37 A city borrows \$500 000 and agrees to pay interest semiannually at $j_2 = 10\%$. (*a*) What semiannual deposits must be made into a sinking fund earning interest at $j_2 = 6\%$ in order to repay the debt in 15 years? (*b*) What is the total semiannual expense of the debt?

(*a*) To find the semiannual sinking-fund deposit R, apply (5.1), with $S = 500\ 000$, $i = 0.03$, $n = 30$:

$$R = \frac{500\ 000}{s_{\overline{30}|.03}} = \$10\ 509.63$$

(*b*) Interest on the debt is $500\ 000(0.05) = \$25\ 000$ semiannually. The total semiannual expense is then

$$25\ 000 + 10\ 509.63 = \$35\ 509.63$$

7.38 For Problem 7.37, find the book value of the city's indebtedness after 10 years.

The amount in the sinking fund after 10 years is

$$10\ 509.63 s_{\overline{20}|.03} = \$282\ 397.69$$

and so the book value is $500\ 000 - 282\ 397.69 = \$217\ 602.31$.

7.6 COMPARISON OF AMORTIZATION AND SINKING-FUND METHODS

One can compare the amortization and sinking-fund methods of repaying a debt by comparing the periodic expenses under the two methods.

Let i be the interest rate per period on the debt for both the sinking-fund and the amortization method, and let r be the interest rate for the same period on the sinking fund. Let the term of the loan be n interest conversion periods and let A be the principal on the loan.

The periodic expense, E_1, under the amortization method is, using Problem 5.75(a),

$$E_1 = \frac{A}{a_{\overline{n}|i}} = Ai + \frac{A}{s_{\overline{n}|i}}$$

The periodic expense, E_2, under the sinking-fund method is

$$E_2 = Ai + \frac{A}{s_{\overline{n}|r}}$$

Thus, if $i > r$, then $s_{\overline{n}|i} > s_{\overline{n}|r}$ and $E_1 < E_2$ (the amortization method is preferable); if $i = r$, then $E_1 = E_2$; and if $i < r$, then $E_1 > E_2$ (the sinking-fund method is preferable).

When the interest rate on the debt under the amortization method is different from the interest rate on the debt under the sinking-fund method, the above conclusions do not apply. Instead we must calculate and directly compare E_1 and E_2.

SOLVED PROBLEMS

7.39 A company can borrow \$200 000 for 15 years. They can amortize the debt at $j_1 = 11\%$ or they can pay interest on the loan at $j_1 = 10.5\%$ and set up a sinking fund at $j_1 = 7\frac{1}{2}\%$ to repay the principal in 15 years. Which plan is cheaper, and by how much per annum?

Under the amortization method:

$$E_1 = \frac{200\ 000}{a_{\overline{15}|.11}} = \$27\ 813.05$$

Under the sinking-fund method, the annual interest expense is $200\ 000(0.105) = \$21\ 000$ and the annual sinking-fund deposit required is

$$\frac{200\ 000}{s_{\overline{15}|.075}} = \$7657.45$$

for a total annual expense $E_2 = \$28\ 657.45$. Thus the amortization method is better by \$844.40 a year.

7.40 A company can borrow \$100 000 for 10 years by paying interest as it falls due at $j_2 = 14\%$ and by setting up a sinking fund at $j_{12} = 12\%$ that would require semiannual deposits. At what rate j_2 would an amortization method have the same semiannual expense?

Under the sinking-fund method, the semiannual interest payment is $100\ 000(0.07) = \$7000$. To find the semiannual sinking-fund deposit, first find the rate i per half-year such that

$$\begin{aligned}(1+i)^2 &= (1.01)^{12}\\ i &= (1.01)^6 - 1 = 0.061520151\end{aligned}$$

Hence, the semiannual sinking fund deposit is

$$\frac{100\ 000}{s_{\overline{20}|.061520151}} = \$2674.34$$

for a total semiannual expense of \$9674.34.

Now we find the rate of interest $j_2 = 2i$ such that

$$100\ 000 - 9674.34a_{\overline{20}|i} \qquad \text{or} \qquad a_{\overline{20}|i} = 10.3366$$

Using linear interpolation to find the rate j_2 (see Section 5.6), we calculate:

$a_{\overline{20}\vert i}$	j_2
10.5940	14%
10.3366	j_2
10.1945	15%

Differences: 0.2574 (between 10.5940 and 10.3366), corresponding to x; 0.3995 (between 10.5940 and 10.1945), corresponding to 1%.

$$\frac{x}{1\%} = \frac{0.2574}{0.3995}$$

$$x = 0.64\%$$

$$j_2 = 14.64\%$$

7.41 **A firm can borrow \$350 000 at $j_{12} = 12\%$ and amortize the debt with annual payments over 10 years. From a second source, the money can be borrowed at $j_1 = 11\frac{3}{4}\%$ if the interest is paid annually and annual deposits are made into a sinking fund to repay the \$350 000 in 10 years. What rate, j_4, must the sinking fund earn for the annual expense to be the same under the two options?**

To find the annual expense E_1 of the amortization method, we find the rate i per year such that

$$1 + i = (1.01)^{12} \qquad \text{or} \qquad i = 0.12682503$$

Then

$$E_1 = \frac{350\ 000}{a_{\overline{10}|.12682503}} = \$63\ 684.98$$

Under the sinking-fund method, the annual interest payment is 350 000(0.1175) = \$41 125, leaving

$$63\ 684.98 - 41\ 125.00 = \$22\ 559.98$$

for the annual sinking-fund deposit. Now we want to find the sinking-fund rate i per half-year such that

$$\begin{aligned} 22\ 559.98s_{\overline{10}|i} &= 350\ 000 \\ s_{\overline{10}|i} &= 15.5142 \end{aligned}$$

Solving by linear interpolation,

$s_{\overline{10}\vert i}$	i
15.1929	9%
15.5142	i
15.9374	10%

Differences: 0.3213 (between 15.1929 and 15.5142), corresponding to x; 0.7445 (between 15.1929 and 15.9374), corresponding to 1%.

$$\frac{x}{1\%} = \frac{0.3213}{0.7445}$$

$$x = 0.43\%$$

$$i = 9.43\%$$

Finally, we determine j_4 such that

$$\begin{aligned} \left(1 + \frac{j_4}{4}\right)^4 &= 1.0943 \\ j_4 &= 4[(1.0943)^{1/4} - 1] = 9.11\% \end{aligned}$$

Supplementary Problems

AMORTIZATION OF A DEBT

7.42 A loan of \$20 000 is to be amortized with equal monthly payments over a 3-year period at $j_{12} = 8\%$. Find the concluding payment, if the monthly payment is rounded up to (*a*) the cent, (*b*) the dollar. *Ans.* (*a*) \$626.62; (*b*) \$623.85

7.43 A \$4000 loan is to be amortized with eight equal quarterly payments over 2 years. If interest is at $j_4 = 10\%$, (*a*) find the quarterly payment, and (*b*) construct an amortization schedule. *Ans.* (*a*) \$557.87

7.44 A debt of \$1000, bearing interest at $j_{12} = 13\frac{1}{2}\%$, is amortized by monthly payments of \$200 for as long as necessary. Construct the amortization schedule.

7.45 A debt of \$2000 will be repaid by monthly payments of \$500 for as long as necessary, the first payment to be made at the end of 6 months. If interest is at $j_{12} = 9\%$, find the size of the debt at the end of 5 months and make out the complete schedule starting at that time. *Ans.* \$2076.13

7.46 A recreational vehicle worth \$46 000 is purchased with a down payment of \$6000 and monthly payments for 15 years. If interest is $j_2 = 10\%$, (*a*) find the monthly payment required, and (*b*) complete the first six lines of the amortization schedule. *Ans.* (*a*) \$424.91

7.47 A couple purchases a house worth \$116 000 by paying \$16 000 down and then taking out a mortgage at $j_{12} = 8\%$ to be amortized over 25 years with equal monthly payments. (*a*) Find the monthly payment. (*b*) Make a partial amortization schedule showing the distribution of the first 6 payments as to principal and interest. *Ans.* (*a*) \$771.82

7.48 Redo Problem 7.47, using (i) $j_{12} = 6\%$, (ii) $j_{12} = 10\%$. *Ans.* (i) \$644.31, (ii) \$908.71

7.49 A loan is being repaid over 10 years in equal annual installments. If the amount of principal in the third payment is \$350, find the principal portion of the eighth payment, given (*a*) $j_1 = 8\frac{1}{2}\%$; (*b*) $j_{12} = 8\frac{1}{2}\%$. *Ans.* (*a*) \$526.28; (*b*) \$534.56

7.50 The Andersons borrow \$8000 to be repaid in monthly installments over 4 years at $j_{12} = 13\frac{1}{2}\%$. Find the total amount of interest paid over the 4 years. *Ans.* \$2397.31

7.51 A loan is being repaid with 10 annual installments. The principal portion of the seventh payment is \$110.25 and the interest portion is \$39.75. What annual effective rate of interest is being charged? *Ans.* 8%

7.52 A loan at $j_1 = 9\%$ is being repaid by monthly payments of \$750 each. The total principal repaid in the 12 monthly installments of the 8th year is \$400. What is the total interest paid in the 12 installments of the 10th year? *Ans.* \$8524.76

7.53 You lend a friend \$15 000 to be amortized by semiannual payments for 8 years, with interest at $j_2 = 9\%$. You deposit each payment in an account paying $j_{12} = 7\%$. What annual effective rate of interest have you earned over the entire 8-year period? *Ans.* 8.17%

7.54 A loan of \$10 000 is being repaid by semiannual payments of \$1000 on account of principal. Interest on the outstanding balance at j_2 is paid in addition to the principal repayments. The total of all payments is \$12 200. Find j_2. *Ans.* 8%

7.55 A loan of \$$A$ is to be repaid by 16 equal semiannual installments, including principal and interest, at rate i per half year. The principal in the first installment (6 months hence) is \$30.83. The principal in the last is \$100.00. Find the annual effective rate of interest. *Ans.* 16.99%

7.56 A loan is to be repaid by 16 quarterly payments of \$50, \$100, \$150, ..., \$800, the first payment due 3 months after the loan is made. Interest is at a nominal annual rate of 8% compounded quarterly. Find the total amount of interest contained in the payments. *Ans.* \$1314.67

7.69 A 5-year loan is being repaid with level monthly installments at the end of each month, beginning with January 1993 and continuing through December 1997. A 12% nominal annual interest rate compounded monthly was used to determine the amount of each monthly installment. On which date will the outstanding principal of this loan first fall below one-half of the original amount of the loan? *Ans.* November 1, 1995

7.70 A debt is amortized at $j_4 = 10\%$ by payments of \$300 per quarter. If the outstanding principal is \$2853.17 just after the kth payment, what was it just after the $(k-1)$st payment? *Ans.* \$3076.26

7.71 A loan is made on January 1, 1975, and is to be repaid by 25 level annual installments. These installments are in the amount of \$3000 each and are payable on December 31 of the years 1975 through 1999. However, just after the December 31, 1979, installment has been paid, it is agreed that, instead of continuing the annual installments on the basis just described, henceforth installments will be payable quarterly with the first such quarterly installment being payable on March 31, 1980, and the last one on December 31, 1999. Interest is at an annual effective rate of 10%. By changing from the old repayment schedule to the new one, the borrower will reduce the total amount of payments made over the 25-year period. Find the amount of this reduction. *Ans.* \$2126.40

MORTGAGES

7.72 Mrs. Adams buys a house, taking a \$60 000, 20-year mortgage at $j_{12} = 13\frac{1}{2}\%$ from a lender who charges 5 points. Find the true rate of interest, j_{12}, being charged. *Ans.* 14.39%

7.73 If, in Problem 7.72, Mrs. Adams pays off the outstanding balance of the mortgage after 12 years, what is the true rate of interest, j_{12}? *Ans.* 14.42%

7.74 ABC Savings and Loans develops a special scheme to help their customers pay their mortgages off quickly. Instead of making payments of $\$X$ once a month, mortgage borrowers are asked to pay $\$X/4$ once a week (52 times a year). The Cohens are buying a house and need a \$95 000 mortgage at $j_{12} = 9\%$. Determine (*a*) the monthly payment required to amortize the debt over 25 years, (*b*) the weekly payment as suggested in the scheme, (*c*) the number of weeks it will take to pay off the loan using the weekly schedule, (*d*) the interest saved using the weekly schedule. *Ans.* (*a*) \$797.24; (*b*) \$199.31; (*c*) 1003; (*d*) \$39 280.92

7.75 Mr. Ramsay can buy a certain house for \$190 000 if he takes out a \$160 000 mortgage from a bank at $j_{12} = 12\%$. The loan would be amortized over 25 years, but the rate of interest would be fixed only for 5 years, after which the loan would be renegotiated. The seller of the house is willing to give Mr. Ramsay a mortgage at $j_{12} = 10.5\%$. The monthly payment would be determined using a 25-year repayment schedule. The seller would guarantee the rate of interest for 5 years, at which time Mr. Ramsay would have to renegotiate the outstanding principal. If Mr. Ramsay accepts this offer, the seller will want \$200 000 for the house, forcing Mr. Ramsay to borrow \$170 000. If Mr. Ramsay can earn $j_{12} = 6\%$ on his savings, what should he do? *Ans.* Pay \$190 000

7.76 The Hwangs buy a home and assume a \$90 000 mortgage to be amortized with monthly payments over 20 years at $j_{12} = 9\%$. The mortgage contract allows the Hwangs to make extra payments of principal each month. If they can pay an extra \$100 each month, how long will it take to repay the mortgage, and what will be the size of the final smaller payment? *Ans.* 182 months, \$266.43

7.77 Tristar Corporation built a new plant in 1992 at a cost of \$1 700 000. It paid \$200 000 cash and assumed a mortgage for \$1 500 000 to be repaid over 10 years by equal semiannual payments due each June 30 and December 31, the first payment being due on December 31, 1992. The mortgage interest rate is 11% per annum compounded semiannually and the original date of the loan was July 1, 1992.

(*a*) What will be the total of the payments made in 1994 on this mortgage?

(b) Mortgage interest paid in any year (for this mortgage) is an income tax deduction for that year. What will be the interest deduction on Tristar Corporation's 1994 tax form?

(c) Suppose the plant is sold on January 1, 1996. The buyer pays \$650 000 cash and assumes the outstanding mortgage. What is Tristar Corporation's capital gain (amount realized less original price) on the investment in the building?

Ans. (a) \$251 038; (b) \$147 230.30; (c) \$94 366.50

7.78 You are choosing between two mortgages for \$60 000 with a 20-year amortization period. Both charge $j_{12} = 10.5\%$ and permit the mortgage to be paid off in less than 20 years. Mortgage A allows you to make weekly payments, with each payment being $\frac{1}{4}$ of the normal monthly payment. Mortgage B allows you to make double the usual monthly payment every 6 months. Assuming that you will take advantage of the mortgage provisions, calculate the total interest charges over the life of each mortgage to determine which mortgage costs less.
Ans. Mortgage B

7.79 The Smiths buy a home and take out an \$160 000 mortgage on which the interest rate is allowed to float freely. At the time the mortgage is issued, interest rates are $j_2 = 10\%$ and the Smiths choose a 25-year amortization schedule. Six months into the mortgage, interest rates rise to $j_2 = 12\%$. Three years into the mortgage (after 36 payments) interest rates drop to $j_2 = 11\%$ and 4 years into the mortgage, interest rates drop to $j_2 = 9\frac{1}{2}\%$. Find the outstanding balance of the mortgage after 5 years. (The monthly payment is set at issue and does not change.)
Ans. \$161 937.39

REFINANCING A LOAN

7.80 A borrower is repaying an \$8000 loan at $j_{12} = 15\%$ with monthly payments over 3 years. Just after the twelfth payment (at the end of 1 year), he has the balance refinanced at $j_{12} = 12\%$. If the number of payments remains unchanged, what will be the new monthly payment, and what will be the total savings in interest? *Ans.* \$269.24, \$194.00

7.81 The Smiths buy a refrigerator and stove for \$1400 in total. They finance the purchase at $j_{12} = 15\%$ to be repaid over 36 months (3 years). If they wish to pay off the loan early, they will incur a penalty equal to three times one month's interest on the effective outstanding balance. After 12 payments, they notice that interest rates at their local Credit Union are $j_{12} = 11\%$. Should they refinance? *Ans.* No

7.82 Mrs. Dent buys \$5000 worth of home furnishings from the ABC Furniture Mart. She pays \$500 down and agrees to pay the balance in monthly installments over 5 years at $j_{12} = 18\%$. The contract stipulates that in case of early repayment there is a penalty equal to three months' payments. After 2 years (24 payments), Mrs. Dent realizes that she can borrow the money from the bank at $j_{12} = 12\%$. Should she refinance? *Ans.* No

7.83 A couple buy a house and take out a \$50 000 mortgage to be repaid with monthly payments over 20 years at $j_{12} = 9.6\%$. After $3\frac{1}{2}$ years they sell their house and pay off the mortgage. They find that in addition to repaying the loan balance, they must pay a penalty equal to three times one month's interest on the outstanding balance. What total amount must they repay?
Ans. \$47 672.25

7.84 Vera is repaying a \$5000 loan at $j_{12} = 16\frac{1}{2}\%$ with monthly installments over 3 years. Owing to temporary unemployment, she misses the 13th through 18th payment, inclusive. Find the value of the revised monthly payment needed, starting in the 19th month, if the loan is still to be repaid at $j_{12} = 16\frac{1}{2}\%$ by the end of the original 3 years. *Ans.* \$246.38

7.85 A loan effective January 1, 1994 is being amortized by equal monthly installments over 5 years using interest at a nominal annual rate of 12% compounded monthly. The first such installment was due February 1, 1994, and the last such installment was to be due January 1, 1999. Immediately after the 24th installment was made on January 1, 1996, a new level monthly installment is determined (using the same rate of interest) in order to shorten the total amortization period

7.57 A loan of \$20 000 with interest at $j_{12} = 15\%$ is amortized by equal monthly payments over 15 years. In which payment will the interest portion be less than the principal portion, for the first time? *Ans.* 126th

OUTSTANDING PRINCIPAL

7.58 A loan is being repaid by monthly installments of \$250 at $j_{12} = 8\%$. If the loan balance after the fourth monthly payment is \$2800, find the original loan value. *Ans.* \$3710.11

7.59 Below is part of an amortization schedule based on monthly payments:

Payment Number	Distribution of Payment	
	Interest	Principal
k	39.64	92.03
$k+1$	38.72	92.95

Determine (*a*) the monthly payment, (*b*) the effective rate of interest per month (closest $\frac{1}{8}\%$), (*c*) the nominal rate of interest j_{12} (closest $\frac{1}{8}\%$). (*d*) Using the rate calculated in (*b*), find the outstanding balance just after payment k and find the remaining period of the loan beyond the date of payment $k+1$. *Ans.* (*a*) \$131.67; (*b*) 1%; (*c*) 12%; (*d*) \$3872.00, 34 months

7.60 To pay off the purchase of a car, Chantal got a \$15 000, 4-year bank loan at $j_{12} = 9\%$. She makes monthly payments. How much does she still owe on the loan at the end of 2 years (24 payments)? Use both the retrospective and prospective methods. *Ans.* \$8170.57

7.61 On July 1, 1995, Brian borrowed \$30 000 to be repaid over 3 years with monthly payments at $j_{12} = 8\%$ (first payment August 1, 1995). How much principal was repaid in 1995? How much interest? *Ans.* \$3750.17, \$950.33

7.62 A doctor buys a house worth \$380 000 by paying \$125 000 down and then taking a mortgage loan out for \$255 000. The mortgage is at $j_{12} = 7\frac{1}{2}\%$ and will be repaid over 20 years in monthly installments. How much of the debt does the doctor pay off in the first year? *Ans.* \$5720.22

7.63 To pay off the purchase of home furnishings, a couple takes out a bank loan of \$3000 to be retired with monthly payments over 2 years at $j_\infty = 8\%$. (*a*) What is the outstanding debt just after the tenth payment? (*b*) What is the principal portion of the eleventh payment?
Ans. (*a*) \$1808.03; (*b*) \$123.63

7.64 Martha buys a piece of land worth \$40 000 by paying \$10 000 down and then taking out a loan for \$30 000. The loan will be retired with quarterly payments over 15 years at $j_4 = 14\%$. Find her equity at the end of 9 years. *Ans.* \$20 687.36

7.65 A loan of \$15 000 is being repaid by installments of \$350 at the end of each month for as long as necessary, plus a final smaller payment. If interest is at $j_4 = 10\%$, find the outstanding balance at the end of 2 years. *Ans.* \$9027.10

7.66 The Smiths borrow \$7500 to buy a car. The loan will be repaid over 4 years with monthly payments at $j_{12} = 15\%$. Find the total interest paid in the 12 payments of year 3.
Ans. \$512.40

7.67 A loan of \$2000 is to be repaid by annual payments of \$400 per annum for the first 5 years and payments of \$450 per year thereafter for as long as necessary. Find the total number of payments and the amount of the smaller final payment made 1 year after the last regular payment. Assume annual effective rate of 18%. *Ans.* 12 payments, \$445.69

7.68 Five years ago, Justin deposited \$1000 into a fund out of which he draws \$100 at the end of each year. The fund guarantees interest at 5% on the principal on deposit during the year. If the fund actually earns interest at a rate in excess of 5%, the excess interest earned during the year is paid to Justin at the end of the year in addition to the regular \$100 payment. The fund has been earning 8% each year for the past 5 years. What is the total payment Justin now receives?
Ans. \$123.54

to $3\frac{1}{2}$ years, so the final installment will fall due on July 1, 1997. Find the ratio of the new monthly installment to the original monthly installment. *Ans.* 1.836017314

7.86 A loan of \$50 000 was being repaid by monthly level installments over 20 years at $j_{12} = 9\%$ interest. Now, when 10 years of the repayment period are still to run, it is proposed to increase the interest rate to $j_{12} = 10\frac{1}{2}\%$. What should the new level payment be so as to liquidate the loan on its original due date? *Ans.* \$479.18

7.87 The Mosers buy a camper trailer and take out a \$15 000 loan. The loan is amortized over 10 years with monthly payments at $j_2 = 18\%$.

(*a*) Find the monthly payment needed to amortize this loan.

(*b*) Find the amount of interest paid by the first 36 payments.

(*c*) After 3 years (36 payments) they could refinance their loan at $j_2 = 16\%$ provided they pay a penalty equal to 3 months' interest on the outstanding balance. Should they refinance? Show the difference in their monthly payments.

Ans. (*a*) \$264.13; (*b*) \$7302.74; (*c*) Yes, \$2.83

7.88 A couple buys a home and signs a mortgage contract for \$120 000 to be paid with monthly payments over a 25-year period at $j_2 = 10\frac{1}{2}\%$. After 5 years, they renegotiate the interest rate and refinance the loan at $j_2 = 7\%$. Find (*a*) the monthly payment for the initial 5-year period; (*b*) the new monthly payment after 5 years; (*c*) the accumulated value of the savings for the second 5-year period at $j_{12} = 3\%$ valued at the end of the second 5-year period; (*d*) the outstanding balance at the end of 10 years.
Ans. (*a*) \$1114; (*b*) \$113 271.22; (*c*) \$15 682.65; (*d*) \$97 554.61

7.89 Give the repayment schedule for Problem 7.43, using the sum-of-digits method.

7.90 Give the repayment schedule for Problem 7.44, using the sum-of-digits method.

7.91 Redo Problem 7.46, using the sum-of-digits method.

7.92 Redo Problem 7.47, using the sum-of-digits method.

7.93 To pay off the purchase of a car, a woman gets a \$6000, 3-year bank loan at $j_{12} = 18\%$ requiring monthly payments. Find the outstanding balance on the loan after the 24th payment, using the sum-of-digits method. *Ans.* \$2390.98

7.94 Redo Problem 7.80, using the sum-of-digits method to determine the principal that is refinanced at the end of 1 year. *Ans.* \$271.25, \$145.81

7.95 A loan of \$20 000 is to be retired with monthly payments over 10 years at $j_{12} = 15\%$. Using the sum-of-digits method, (*a*) construct the first two lines of the repayment schedule; (*b*) find the interest and principal portions of the 10th payment; (*c*) find the outstanding balance at the end of 3 years and compare it with the outstanding balance at the same time under the amortization method; (*d*) decide whether the loan should be refinanced at the end of 3 years at $j_{12} = 12\%$, with the term of the loan unchanged.
Ans. (*b*) \$286.22, \$36.45; (*c*) \$17 898.79, \$16 721.46; (*d*) yes

7.96 Consider a \$10 000 loan being repaid with monthly payments over 15 years at $j_{12} = 15\%$. Find the outstanding balance (*a*) at the end of 2 years and (*b*) at the end of 5 years using both the sum-of-digits method and the amortization method.
Ans. (*a*) \$10 412.52, \$9584.29; (*b*) \$10 024.06, \$8674.92

7.97 Michelle has a \$5000 loan that is being repaid by monthly payments over 4 years at $j_{12} = 15\%$. The lender uses the sum-of-digits method to determine outstanding balances. After 1 year of payments the lender's interest rate on new loans has dropped to $j_{12} = 12\%$. Will Michelle save money by refinancing the loan if the term of the loan remains the same?
Ans. Yes, \$4.36 a month

7.98 Matthew can borrow \$15 000 at $j_4 = 15\%$ and repay the loan with monthly payments over 10 years. If he wants to pay the loan off early, the outstanding balance will be determined using the sum-of-digits method. He can also borrow \$15 000 with monthly payments over 10 years at $j_4 = 16\%$ and pay the loan off at any time without penalty. The outstanding balance will be determined using the amortization method. Matthew has an endowment insurance policy coming due in 4 years that could be used to pay off the outstanding balance at that time in full. Which loan should he take if he earns $j_{12} = 6\%$ on his savings? *Ans.* Borrow at $j_4 = 16\%$

7.99 A loan of \$18 000 is to be repaid with monthly payments over 10 years at $j_2 = 17\frac{1}{2}\%$. Using the sum-of-digits method (*a*) construct the first two and the last two lines of the repayment schedule; (*b*) find the interest and the principal portion of the 10th payment; (*c*) find the outstanding balance at the end of 2 years and compare it with the outstanding balance at the same time calculated by the amortization method; (*d*) advise whether the loan should be refinanced at the end of 2 years at current rate $j_2 = 16\%$ with the term of the loan unchanged.
Ans. (*b*) \$296.54, \$15.09; (*c*) \$17 477.61, \$16 351.36; (*d*) Do not refinance

7.100 A \$20 000 home renovation loan is to be amortized over 10 years by monthly payments, with each regular payment rounded up to the next dollar, and the last payment reduced accordingly. Interest on the loan is at $j_4 = 10\%$. After 4 years the loan is fully paid off with an extra payment. Find the amount of this final payment if the sum-of-digits method is used to calculate the outstanding principal. *Ans.* \$14 700.94

SINKING FUNDS

7.101 A couple is saving a down payment for a house. They want to have \$15 000 at the end of 4 years in an account that pays interest at $j_1 = 6\%$. (*a*) How much must be deposited in the fund at the end of each year? (*b*) Construct a complete sinking-fund schedule. *Ans.* (*a*) \$3428.87

7.102 A condominium high rise consists of 128 two-bedroom units of uniform size. The Owners' Association establishes a fund to save \$130 000 in 5 years to install a gymnasium. Assuming that the association can earn $j_{12} = 7\frac{1}{2}\%$ on its money, (*a*) what monthly sinking-fund assessment will be required per unit? (*b*) Show the first three and last two lines of the sinking-fund schedule.
Ans. (*a*) \$14

7.103 A sinking fund earning interest at $j_\infty = 13\%$ now contains \$2000. (*a*) What quarterly deposits for the next 3 years will cause the fund to grow to \$10 000? (*b*) How much is in the fund 2 years from now? *Ans.* (*a*) \$487.98; (*b*) \$6980.14

7.104 A couple wants to save \$20 000 to buy some land. They can save \$150 a month in an account paying interest at $j_4 = 10\%$. How many deposits will be required, and what will be the size of the final smaller deposit? *Ans.* 91, \$0

7.105 A cottagers' association decides to set up a sinking fund to save enough money to have their cottage road widened. They need \$30 000 at the end of 5 years in a fund earning $9\frac{1}{2}\%$ per annum. What annual deposit is required per cottage, if there are 40 cottages on the road?
Ans. \$124.08

7.106 A homeowners' association decided to set up a sinking fund to accumulate \$50 000 by the end of 3 years to improve recreational facilities. (*a*) What monthly deposits are required if the fund earns 5% compounded daily? (*b*) Show the first three and the last two lines of the sinking fund schedule. *Ans.* (*a*) \$1290.02

7.107 Consider an amount that is to be accumulated with equal deposits R at the end of each interest period for 5 periods at rate i per period. Hence, the amount to be accumulated is $Rs_{\overline{5}|i}$. Do a complete schedule for this sinking fund. Verify that the sum of the interest column plus the sum of the deposit-column equals the sum of the increase-in-the-fund column, and both sums equal the final amount in the fund.

7.108 In its manufacturing process, a company uses a machine that costs $75 000 and is scrapped at the end of 15 years with a value of $5000. The company sets up a sinking fund to finance the replacement of the machine, assuming no change in price, with level payments at the end of each year. Money can be invested at an annual effective interest rate of 4%. Find the value of the sinking fund at the end of the 10th year. *Ans.* $41 971.91

7.109 A sinking fund is being accumulated at $j_{12} = 6\%$ by deposits of $200 per month. If the fund contains $5394.69 just after the kth deposit, what did it contain just after the $(k-1)$st deposit? *Ans.* $5168.85

7.110 A man is repaying a $16 000 loan by the sinking-fund method. His total monthly expense is $350. Out of this $350, interest is paid monthly at $j_{12} = 12\%$ and the remainder of the money is deposited in a sinking fund earning interest at $j_{12} = 10\frac{1}{2}\%$. Find the duration of the loan and the final smaller payment. *Ans.* 64 months, $0

7.111 A borrower of $5000 agrees to pay interest semiannually at $j_2 = 10\%$ on the loan and to build up a sinking fund, which will repay the loan at the end of 5 years. If the sinking fund accumulates at $j_2 = 7\%$, (*a*) find his total semiannual expense. (*b*) How much is in the sinking fund at the end of 4 years? *Ans.* (*a*) $676.21; (*b*) $3857.92

7.112 A city borrows $250 000, paying interest annually on this sum at $j_1 = 9\frac{1}{2}\%$. An annual deposit must be made into a sinking fund earning interest at $j_1 = 6\%$ in order to pay off the entire principal at the end of 15 years. What is the total annual expense of the debt? *Ans.* $34 490.69

7.113 A company issues $500 000 worth of bonds, paying interest at $j_2 = 12\%$. A sinking fund with semiannual deposits accumulating at $j_2 = 9\%$ is established to redeem the bonds at the end of 20 years. Find (*a*) the semiannual expense of the debt, (*b*) the book value of the company's indebtedness at the end of the 15th year. *Ans.* (*a*) $34 671.57; (*b*) $215 001.20

7.114 On a debt of $10 000, interest is paid semiannually at $j_2 = 10\%$ and semiannual deposits are made into a sinking fund to retire the debt at the end of 5 years. If the sinking fund earns interest at $j_{12} = 6\%$, what is the semiannual expense of the debt? *Ans.* $1370.79

7.115 A 10-year loan of $10 000 at $j_1 = 11\%$ is to be repaid by the sinking-fund method, with interest and sinking-fund payments made at the end of each year. The rate of interest earned in the sinking fund is $j_1 = 5\%$. Immediately after the fifth year's payment, the lender requests that the outstanding principal be repaid in one lump sum. Calculate the amount of extra cash the borrower has to raise in order to extinguish the debt. *Ans.* $5606.85

7.116 Interest at $j_2 = 12\%$ on a loan of $3000 must be paid semiannually as it falls due. A sinking fund accumulating at $j_4 = 8\%$ is established to enable the debtor to repay the loan at the end of 4 years. (*a*) Find the semiannual sinking-fund deposit and construct the last two lines of the sinking-fund schedule, based on semiannual deposits. (*b*) Find the semiannual expense of the loan. (*c*) What is the outstanding principal (book value of the loan) at the end of 2 years? *Ans.* (*a*) $325.12; (*b*) $505.12; (*c*) $1618.57

7.117 Mr. White borrows $15 000 for 10 years. He makes total payments, annually, of $2000. The lender receives $j_1 = 10\%$ on his investment each year for the first 5 years and $j_1 = 8\%$ for the second 5 years. The balance of each payment is invested in a sinking fund earning $j_1 = 7\%$. (*a*) Find the amount by which the sinking fund is short of repaying the loan at the end of 10 years. (*b*) By how much would the sinking-fund deposit (in each of the first 5 years only) need to be increased so that the sinking fund at the end of 10 years will be just sufficient to repay the loan? *Ans.* (*a*) $6366.56; (*b*) $789.34

7.118 A $100 000 loan is to be repaid in 15 years, with a sinking fund accumulated to repay principal plus interest. The loan charges $j_2 = 12\%$, while the sinking fund earns $j_2 = 9\%$. What semiannual sinking-fund deposit is required? *Ans.* $9414.47

7.119 A loan of \$20 000 bears interest on the amount outstanding at $j_1 = 10\%$. A deposit is to be made in a sinking fund earning interest at $j_1 = 4\%$, which will accumulate enough to pay one-half of the principal at the end of 10 years. In addition, the debtor will make level payments to the creditor, which will pay interest at $j_1 = 10\%$ on the outstanding balance first and the remainder will repay the principal. What is the total annual payment, including that made to the creditor and that deposited in the sinking fund, if the loan is to be completely retired at the end of 10 years? *Ans.* \$4460.37

7.120 John borrows \$10 000 for 10 years and uses a sinking fund to repay the principal. The sinking-fund deposits earn an annual effective interest rate of 5%. The total required payment for both the interest and the sinking-fund deposit made at the end of each year is \$1445.05. Calculate the annual effective interest rate charged on the loan. *Ans.* 6.5%

7.121 A company borrows \$10 000 for 5 years. Interest of \$600 is paid semiannually. To repay the principal of the loan at the end of 5 years, equal semiannual deposits are made into sinking fund that credits interest at a nominal rate of 8% compounded quarterly. The first payment is due in 6 months. Calculate the annual effective rate of interest that the company is paying to service and retire the debt. *Ans.* 14.74%

7.122 On August 1, 1988, Mrs. Chan borrows \$20 000 for 10 years. Interest at 11% per annum convertible semiannually must be paid as it falls due. The principal is replaced by means of level deposits on February 1 and August 1 in years 1989 to 1998 (inclusive) into a sinking fund earning $j_1 = 7\%$ in 1989 through December 31, 1993, and $j_1 = 6\%$ January 1, 1994 through 1998. (*a*) Find the semiannual expense of the loan. (*b*) How much is in the sinking fund just after the August 1, 1997, deposit? (*c*) Show the sinking-fund schedule entries at February 1, 1998, and August 1, 1998.
Ans. (*a*) \$846.42; (*b*) \$17 458.03

COMPARISON OF AMORTIZATION AND SINKING-FUND METHODS

7.123 A company borrows \$50 000 to be repaid in equal installments at the end of each year for 10 years. Find the total annual cost under the following conditions:

(*a*) The debt is amortized at $j_1 = 9\%$

(*b*) Interest at 9% is paid on the debt and a sinking fund is set up at $j_1 = 9\%$

(*c*) Interest at 9% is paid on the debt and a sinking fund is set up at $j_1 = 6\%$

Ans. (*a*) \$7791; (*b*) \$7791; (*c*) \$8293.40

7.124 A company can borrow \$180 000 to be repaid over 15 years. They can amortize the debt at $j_1 = 10\%$ or they can pay interest on the loan at $j_1 = 9\%$ and set up a sinking fund at $j_1 = 7\%$. Which plan is cheaper, and by how much per annum? *Ans.* Sinking fund, \$302.25

7.125 A firm borrows \$60 000 to be repaid over 5 years. One source will lend them the money at $j_2 = 10\%$ if it is amortized by semiannual payments. A second source will lend them the money at $j_2 = 9.5\%$ if only the interest is paid semiannually and the principal is returned in a lump sum at the end of 5 years. Which source is cheaper, and how much will be saved each half-year if the required sinking fund earns interest at $j_2 = 8\%$? *Ans.* First source, \$77.19

7.126 A city can borrow \$500 000 for 20 years by issuing bonds on which interest will be paid semiannually at $j_2 = 9\frac{1}{8}\%$. The principal will be paid off by a sinking fund consisting of semiannual deposits invested at $j_2 = 8\%$. Find the nominal rate j_2 at which the loan could be amortized at the same semiannual cost. *Ans.* 9.42%

7.127 A firm can borrow \$200 000 at $j_1 = 9\%$ and amortize the debt for 10 years. From a second source, the money can be borrowed at $j_1 = 8\frac{1}{2}\%$ if the interest is paid annually and the principal is repaid in a lump sum at the end of 10 years. What yearly rate j_1 must the sinking fund earn for the annual expense to be the same under the two options? *Ans.* 7.45%

7.128 A company wants to borrow \$500 000. One source of funds will agree to lend the money at $j_4 = 8\%$ if interest is paid quarterly and the principal is paid in a lump sum at the end of 15 years. The firm can set up a sinking fund at $j_4 = 6\%$ and will make quarterly deposits. (*a*) What is the total quarterly cost of the loan? (*b*) At what rate j_4 would it be less expensive to amortize the debt over 15 years?
Ans. (*a*) \$15 196.71; (*b*) $j_4 < 8.92\%$

7.129 You are able to repay a \$2000 loan by either (*a*) amortization at $j_{12} = 15\%$ with 12 months payments; or (*b*) at $j_{12} = 13\%$ using a sinking fund earning $j_{12} = 8\%$, and paid off in 1 year. Which method is cheaper? *Ans.* (*a*) is cheaper by \$1.79 a month

7.130 A company needs to borrow \$200 000 for 6 years. One source will lend them the money at $j_2 = 10\%$ if it is amortized by monthly payments. A second source will lend the money at $j_4 = 9\%$ if only the interest is paid monthly and the principal is returned in a lump sum at the end of 6 years. The company can earn interest at $j_{365} = 6\%$ on the sinking fund. Which source should be used for the loan and how much will be saved monthly? *Ans.* First source, \$117.61

7.131 Tanya can borrow \$10 000 by paying the interest on the loan as it falls due at $j_2 = 12\%$ and by setting up a sinking fund with semiannual deposits that accumulate at $j_{12} = 9\%$ over 10 years to repay the debt. At what rate j_4 would an amortization scheme have the same semiannual cost? *Ans.* 13.03%

7.132 A loan of \$10 000 at 16% per annum is to be repaid over 10 years: \$2000 by the amortization method and \$8000 be the sinking-fund method, where the sinking fund can be accumulated with annual deposits at $j_4 = 10\%$. What extra annual payment does the above arrangement require as compared to repayment of the whole loan by the amortization method? *Ans.* \$117.65

7.133 A company wants to borrow a large amount of money for 15 years. One source would lend the money at $j_2 = 9\%$, provided it is amortized over 15 years by monthly payments. The company could also raise the money by issuing bonds paying interest semiannually at $j_2 = 8\frac{1}{2}\%$ and redeemable at par in 15 years. In this case, the company would set up a sinking fund to accumulate the money needed for the redemption of the bonds at the end of 15 years. What rate j_{12} on the sinking fund would make the monthly expense the same under the two options?
Ans. 7.25%

7.134 A \$10 000 loan is being repaid by the sinking-fund method. Total annual outlay (each year) is \$1400 for as long as necessary, plus a smaller final payment made 1 year after the last regular payment. If the lender receives $j_1 = 8\%$ and the sinking fund accumulates at $j_1 = 6\%$, find the time and amount of the last irregular final payment. *Ans.* \$1278.04 at the end of 12 years

7.135 A \$5000 loan can be repaid quarterly for 5 years using amortization and an interest rate of $j_{12} = 10\%$ or by a sinking fund to repay both principal and accumulated interest. If paid by a sinking fund, the interest on the loan will be $j_{12} = 9\%$. What annual effective rate must the sinking fund earn to make the quarterly cost the same for both methods? *Ans.* 8.35%