GROUPS

INTRODUCTION

In mathematics, a **group** is a set provided with an operation that connects any two elements to compose a third element in such a way that the operation is associative, an identity element will be defined, and every element has its inverse. These three conditions are group axioms, hold for number systems and many other mathematical structures. For instance, the set of integers and the addition operation form a group. That means, in simple words, a group is a combination of a set and binary operation.

Before learning in detail about groups, let's understand the basics that are required to define groups in mathematics.

Set: A set is a collection of well-defined things, objects or elements, and it does not vary from person to person and is represented using a capital letter.

Binary operation: An operation that contains two elements of a set given another element of the same set is called a binary composition or binary operation. Generally, a binary operation can be represented using "." or "⋆".

Algebraic structure:

An algebraic structure is a set of elements, i.e., the carrier of the structure with an operation that matches any two members of the set uniquely onto a third member. One of the most basic algebraic structures is the group. The axioms give the specificity of an algebraic structure that it satisfies.

Let's have a look at the mathematics definition of groups.

Group Definition

If G is a non-empty set and "⋆" is the binary operation defined on G such that the following laws or axioms are satisfied then, (G, ⋆) is called a group.

Let "G" be a non-empty set and "⋆" be a binary operation on G such that

(G1) – Closure law	for a, b ∈ G, a ⋆ b ∈ G
(G2) – Associative law	a ⋆ (b ⋆ c) = (a ⋆ b) ⋆ c for all a, b, c ∈ G
(G3) – Identity element	there is an element e ∈ G such that a ⋆ e = e ⋆ a = a for all a ∈ G; where e is the identity element
(G4) – Inverse law	for each a ∈ G, there exists an element b ∈ G such that a ⋆ b = b ⋆ a = e, where b = a^{-1} is the inverse element of a.

Terminology of Groups

We can define various terms related to groups based on the number of laws they satisfy.

Abelian Group or Commutative Group

(G; ⋆) is said to be an abelian group, or a commutative group is a binary operation that satisfies the commutative law, i.e., $a \star b = b \star a$ for all $a, b \in G$.

Semi Group

If the set G satisfies only closure law and associative law, then G is called a semi-closed group or semi group.

Finite and Infinite Group

In a group, G contains only a finite number of elements, then group G is called a finite group; otherwise, group G is called an infinite group.

Order of a group: The number of elements in a finite group G is called the order of a group and is denoted by $O_{(G)}$. That means if the number of elements in G is n, then $O_{(G)} = n$.

Some Examples of Groups:

1. The real numbers with respect to addition, which we denote as $\langle \mathbb{R}, + \rangle$, is a group: it has the identity 0, any element x has an inverse $-x$, and it satisfiesassociativity.
2. Conversely, the real numbers with respect to multiplication, which we denote as $\langle \mathbb{R}, \cdot \rangle$, is **not** a group: the element $0 \in \mathbb{R}$ has no inverse, as there is nothing we can multiply0 by to get to 1!
3. The nonzero real numbers with respect to multiplication, which we denote as $\langle \mathbb{R}^{\times}, \cdot \rangle$,is a group! The identity in this group is 1, every element x has an inverse $1/x$ such that $x \cdot (1/x) = 1$, and this group satisfies associativity.
4. The integers with respect to addition, $\langle \mathbb{Z}, + \rangle$ form a group!
5. The integers with respect to multiplication, $\langle \mathbb{Z}, \cdot \rangle$ do not form a group: for example, there is no integer we can multiply 2 by to get to 1.
6. The natural numbers $\mathbb{N}$ are not a group with respect to either addition or multipli- cation. For example: in addition, there is no element $-1 \in \mathbb{N}$ that we can add to 1to get to 0, and in multiplication there is no natural number we can multiply 2 by to get to 1.
7. Z,Q,R, and C are groups under usual addition.
8. The set of all 2X2 matrices$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \boldsymbol{R}$ is a group under matrix

addition. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the identity element and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ is the inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

9. The Set of all 2X2 non-singular matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \boldsymbol{R}$ is a group under matrix multiplication. We know the matrix multiplication is associative. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity element. The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $|A| = ad - bc \neq 0$.

10. N is not a group under usual addition since there is no element $e \in N$ such that x+e=x.