Group Theory and Puzzles

Amit Kulshrestha
Indian Institute of Science Education and Research, Mohali

October 20 2009

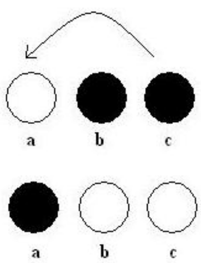
Some puzzles as seen from a group theorists eye

- **Solitaire (Brain Vita) -** Rules of the game match the group law of V_4 (Klein's 4-group).
- **Rubik's Cube -** The game itself is a group (a subgroup of S_{48}).
- **The** 15**-Puzzle -** The game itself is a group (a subgroup of S_{16}).

§1 Solitaire

The game in the beginning looks like:

The rule: A marble eliminates an adjacent marble by jumping over it. One step looks like:



Aim: To eliminate as many marbles as possible.
Information: It is possible to eliminate all but one marble.
Question: Can the last marble remain on any desired location?
Idea: Label the holes with the elements of $V_4 = \{e, a, b, c\}$ and observe some kind of 'law of conservation' while the game is being played.

§2 Rubik's Cube

Consider the following set: $G = \{\text{all possible scrambled positions of Rubik's cube}\}$.
Questions:

- What is the size of G?
- Can one use G to solve Rubik's cube?

Answers:

- $43252003274489856000 = 2^{27}3^{14}5^{3}7^{2}11$
- Yes!

Observations:

- G is a group.
- G is a subgroup of S_{54} (afterall it is all about permuting 9×6 squares - 9 squares on each of the 6 faces).
- G is a subgroup of S_{48} (6 squares - the middle ones on each face - are not being permuted).

Finding generators of G as a subgroup of S_{48}:

			1	2	3						
			4	**T**	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	**L**	13	20	**F**	21	38	**R**	29	36	**Ba**	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	**B**	45						
			46	47	48						

This is how the labelled Rubik's cube looks like when 'opened up'. Rotating the top face by 90° in the clockwise direction is the following permutation in S_{48}:

$T = (1\ 3\ 8\ 6)(2\ 5\ 7\ 4)(9\ 33\ 25\ 17)(10\ 34\ 26\ 18)(11\ 35\ 27\ 19)$ - **T**op

Likewise other 'basic operations' which can be performed on a Rubik's cube are:

- $L = (9\ 11\ 16\ 14)(10\ 13\ 15\ 12)(1\ 17\ 41\ 40)(4\ 20\ 44\ 37)(6\ 22\ 46\ 35)$ - **L**eft
- $F = (17\ 19\ 24\ 22)(18\ 21\ 23\ 20)(6\ 25\ 43\ 16)(7\ 28\ 42\ 13)(8\ 30\ 41\ 11)$ - **F**ront
- $R = (25\ 27\ 32\ 30)(26\ 29\ 31\ 28)(3\ 38\ 43\ 19)(5\ 36\ 45\ 21)(8\ 33\ 48\ 24)$ - **R**ight
- $Ba = (33\ 35\ 40\ 38)(34\ 37\ 39\ 36)(3\ 9\ 46\ 32)(2\ 12\ 47\ 29)(1\ 14\ 48\ 27)$ - **Ba**ck

- $B = (41\ 43\ 48\ 46)(42\ 45\ 47\ 44)(14\ 22\ 30\ 38)(15\ 23\ 31\ 39)(16\ 24\ 32\ 40)$ - **B**ottom

Clearly $G =$ subgroup of S_{48} generated by $\{T, L, F, R, Ba, B\}$.

A solution: Here is how we can solve Rubik's cube. Any scrambled cube is an element, say g of the group G. Solving Rubik's cube now amounts to finding a sequence in $\{T, L, F, R, Ba, B\}$ whose ordered product (in G) is g. To do this, we consider the homomorphism from a free group $\mathcal{F}$ on six generators (say $\{t, l, f, r, p, b\}$ to G given by:

$$t \mapsto T, l \mapsto L, f \mapsto F, r \mapsto R, p \mapsto Ba, b \mapsto B.$$

Under the homomorphism $f : \mathcal{F} \to G$, the inverse image $f^{-1}(g)$ of a scrambled cube g is a coset (of the kernel(f)). Any element in this coset gives a solution of the cube. (For example if $t^2 f p^{-1} l r^2 t^{-1} b \in f^{-1}(g)$ then performing the sequence of operations

$$T \to T \to F^{-1} \to P \to L^{-1} \to R \to R \to T \to B^{-1}$$

will solve the cube). As you can see, there are more than one solutions.

For further reading: Given a set S of generators of a group G and an element $g \in G$, what is the smallest number r such that $\{s_1, s_2, \cdots, s_r\} \in S$ and $g = s_1 s_2 \cdots s_r$? (The number r is called the *length* of g). This takes us to the whole new world of 'Geometric Group Theory', where geometry helps us to understand the length function better!

§3 Food for thought: The 15-Puzzle

Like Rubik's cube, the 15-puzzle is also a permutation puzzle. Try analysing it using group theory.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**14**	**15**	

Symmetric groups and Alternating groups will be useful!