2]. Solve $\frac{d^2y}{dx^2} + y = \csc x$ using method of variation of parameters.

Soln.

Given $(D^2+1)y = \csc x$.

AE $m^2+1=0$

$m^2 = -1$

$m = \pm i$

$CF = C_1 \cos x + C_2 \sin x$

Here $f_1 = \cos x$ | $f_2 = \sin x$

$f_1' = -\sin x$ | $f_2' = \cos x$

Now $f_1 f_2' - f_1' f_2 = \omega$

$\omega = \cos x(\cos x) + \sin x \sin x$

$= \cos^2 x + \sin^2 x$

$\omega = 1$

$PI = Pf_1 + Qf_2$

$P = -\int \frac{f_2 X}{\omega} dx$

$= -\int \frac{\sin x \csc x}{1} dx$

$= -\int \sin x \times \frac{1}{\sin x} dx$

$= -\int dx$

$P = -x$

$Q = \int \frac{f_1 X}{\omega} dx$

$= \int \frac{\cos x \csc x}{1} dx$

$= \int \cos x \times \frac{1}{\sin x} dx = \int \cot x \, dx$

$= \log(\sin x)$

$\therefore PI = -x(\cos x) + \log(\sin x)\sin x$

The general soln. is,

$$y = CF + PI$$
$$= c_1 \cos x + c_2 \sin x - x\cos x + \sin x \log(\sin x)$$

3]. Solve $\frac{d^2y}{dx^2} + y = \cot x$ using method of variation of parameters.

Soln.

Given $(D^2+1)y = \cot x$

AE

$$m^2+1 = 0$$
$$m^2 = -1$$
$$m = \pm i$$

$$CF = c_1 \cos x + c_2 \sin x$$

Here $f_1 = \cos x$ | $f_2 = \sin x$

$f_1' = -\sin x$ | $f_2' = \cos x$

Now, $w = f_1 f_2' - f_1' f_2$

$= 1$

$$PI = Pf_1 + Qf_2$$

Here $P = -\int \frac{f_2 X}{w}\, dx$

$$= -\int \frac{\sin x \cot x}{1}\, dx$$
$$= -\int \sin x \frac{\cos x}{\sin x}\, dx$$
$$= -\int \cos x\, dx$$
$$P = -\sin x$$

and $Q = \int \frac{f_1 X}{w}\, dx = \int \frac{\cos x \cot x}{1}\, dx$

$$= \int \cos x \frac{\cos x}{\sin x}\, dx$$

$$= \int \frac{\cos^2 x}{\sin x}\, dx$$

$$= \int \frac{1-\sin^2 x}{\sin x}\, dx$$

$$= \int [\csc x - \sin x]\, dx$$

$$= \int \csc x\, dx - \int \sin x\, dx$$

$$Q = -\log[\csc x + \cot x] + \cos x$$

$$\therefore PI = -\sin x \cos x + [-\log(\csc x + \cot x) + \cos x]\sin x$$

The general soln. is,

$$y = CF + PI$$

$$= c_1 \cos x + c_2 \sin x - \sin x \cos x + [-\log(\csc x + \cot x)\sin x] + \sin x \cos x$$

$$= c_1 \cos x + c_2 \sin x - \log(\csc x + \cot x)\sin x.$$

4]. Solve $(D^2+a^2)y = \sec ax$ using method of variation of parameters.

Soln.

Given $(D^2+a^2)y = \sec ax$

AE

$$m^2+a^2 = 0$$

$$m^2 = -a^2$$

$$m = \pm ai$$

$$CF = c_1 \cos ax + c_2 \sin ax$$

Here $f_1 = \cos ax$ | $f_2 = \sin ax$.

$f_1' = -a\sin ax$ | $f_2' = a\cos ax$

$$\omega = f_1 f_2' - f_1' f_2$$

$$= \cos ax(a\cos ax) + a\sin ax \sin ax$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$= a[\cos^2 ax + \sin^2 ax] = a(1) = a$$

$$\Rightarrow \omega = a$$

$$PI = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 X}{w}\,dx$$

$$= -\int \frac{\sin ax \ \sec ax}{a}\,dx$$

$$= -\frac{1}{a}\int \sin ax \frac{1}{\cos ax}\,dx = -\frac{1}{a}\int \tan ax\,dx$$

$$= -\frac{1}{a}\frac{\log(\sec ax)}{a} \qquad \int \tan ax = \frac{+\log(\sec ax)}{a}$$

$$P = \frac{-1}{a^2}\log(\sec ax)$$

$$Q = \int \frac{f_1 X}{w}\,dx$$

$$= \int \frac{\cos ax \ \sec ax}{a}\,dx$$

$$= \frac{1}{a}\int \cos ax \frac{1}{\cos ax}\,dx$$

$$= \frac{1}{a}\int dx$$

$$Q = \frac{x}{a}$$

$$PI = -\frac{1}{a^2}\log(\sec ax)\cos ax + \frac{x}{a}\sin ax$$

The general soln. is,

$$y = CF + PI$$

$$= c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\log(\sec ax)\cos ax + \frac{x}{a}\sin ax.$$