Method of variation of Parameters

The second order linear differential eqn. is

$$\frac{d^2 y}{dx^2} + P\frac{dy}{dx} + Q = X \text{ where } X \text{ is a fn. of } x.$$

$CF = C_1 f_1 + C_2 f_2$, C_1, C_2 are constants
where f_1, f_2 are functions of x.

$PI = Pf_1 + Q f_2$

where $P = -\int \frac{f_2 \, X}{f_1 f_2' - f_1' f_2} dx$

$Q = \int \frac{f_2 \, X}{f_1 f_2' - f_1' f_2} dx$

$\int \tan x \, dx = +\log(\sec x)$

$\int \cot x \, dx = \log(\sin x)$

$\int \csc x \, dx = -\log[\csc x + \cot x]$

$\int \sec x \, dx = \log(\sec x + \tan x)$

II. Solve $\dfrac{d^2 y}{dx^2} + 4y = 4\tan 2x$ using method of variation of parameters.

Soln.

Given $(D^2 + 4) y = 4\tan 2x$ where $X = 4\tan 2x$

AE $m^2 + 4 = 0$
            $m^2 = -4$
            $m = \pm 2i$

$CF = C_1 \cos 2x + C_2 \sin 2x$

$PI = Pf_1 + Q f_2$

Here $f_1 = \cos 2x$ $f_2 = \sin 2x$
$f_1' = -2\sin 2x$ $f_2' = 2\cos 2x$

Now $\omega = f_1 f_2' - f_2 f_1'$
$= \cos 2x [2\cos 2x] - \sin 2x (-2\sin 2x)$
$= 2\cos^2 2x + 2\sin^2 2x$
$= 2[\cos^2 2x + \sin^2 2x]$
$= 2(1) = 2$

$$P = -\int \frac{f_2 \, x}{\omega} \, dx$$

$$= -\int \frac{\sin 2x \, 4 \tan 2x}{2} \, dx$$

$$= -2 \int \sin 2x \, \frac{\sin 2x}{\cos 2x} \, dx$$

$$= -2 \int \frac{\sin^2 2x}{\cos 2x} \, dx$$

$$= -2 \int \frac{1 - \cos^2 2x}{\cos 2x} \, dx \qquad = -2 \left[\int \frac{1}{\cos 2x} \, dx - \int \cos 2x \, dx \right]$$

$$= -2 \left[\int \sec 2x \, dx - \int \cos 2x \, dx \right]$$

$$= -2 \left[\frac{\log(\sec 2x + \tan 2x)}{2} + \frac{\sin 2x}{2} \right]$$

$$P = -\log(\sec 2x + \tan 2x) + \sin 2x$$

$$Q = \int \cdot \frac{f_1 \, x}{\omega} \, dx$$

$$= \int \frac{\cos 2x \, 4 \tan 2x}{2} \, dx$$

$$= 2 \int \cos 2x \cdot \frac{\sin 2x}{\cos 2x} \, dx$$

$$= 2 \int \sin 2x \, dx \quad = 2 \left[-\frac{\cos 2x}{2} \right]$$

$$Q = -\cos 2x$$

$$PI = P f_1 + Q f_2$$

$$= \left[-\log(\sec 2x + \tan 2x) + \sin 2x \right] \cos 2x$$

$$\quad - \cos 2x \, \sin 2x$$

$$PI = -\log(\sec 2x + \tan 2x) \cos 2x$$

$$\therefore \ y = CF + PI = c_1 \cos 2x + c_2 \sin 2x - \log(\sec 2x + \tan 2x) \cos 2x$$