Gauss Divergence theorem :

The surface integral of normal component of vector function F over a closed surface S enclosing volume V is equal to the volume integral of divergence of F taking through out the volume V

i.e $$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Verify the gauss divergence theorm (GDT) for $\vec{F} = 4xz\,\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ over the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) + \left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y = 4z - y.$$

$$\nabla\cdot\vec{F} = 4z - y.$$

RHS:

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_0^1\int_0^1\int_0^1 (4z - y)\, dx\, dy\, dz.$$

$$= \int_0^1\int_0^1 (4xz - yx)\Big]_{x=0}^{1} dy\, dz$$

$$= \int_0^1\int_0^1 (4z - y)\, dy\, dz.$$

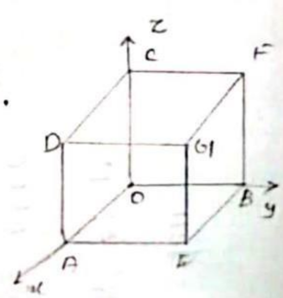
$$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)\Big]_{y=0}^{1} dz.$$

$$= \int_0^1 (4z - 1/2)\, dz.$$

$$= \left[\frac{4z^2}{2} - \frac{1}{2}z\right]_{z=0}^{1}.$$

$$= \frac{4}{2} - \frac{1}{2}.$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = 3/2 \quad \text{——} (1).$$



face	$\hat{n}$	$\vec{F}\cdot\hat{n}$	ds	eqn	$\vec{F}\cdot\hat{n}$ on S	$\iint_S \vec{F}\cdot\hat{n}\,ds$
S_1 AEGD	$\vec{i}$	$4xz$	$dy\,dz$	$x=1$	$4z$	$\int_0^1\int_0^1 4z\,dy\,dz$.
S_2 OBFC.	$-\vec{i}$	$-4xz$.	$dy\,dz$.	$x=0$	0	0
S_3 EBFG	$\vec{j}$	$-y^2$	$dx\,dz$.	$y=1$	-1	$\int_0^1\int_0^1(-1)\,dx\,dz$
S_4 OADC	$-\vec{j}$	$+y^2$	$dx\,dz$	$y=0$	0	$\int$ 0
S_5 DGFC.	$\vec{k}$	yz	$dx\,dy$	$z=1$	y	$\int_0^1\int_0^1 y\,dx\,dy$
S_6 OAEB	$-\vec{k}$	$-yz$	$dx\,dy$	$z=0$	0	0

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} \dots + \iint_{S_2} + \iint_{S_3} \dots + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}.$$

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4z\,dy\,dz + 0.$$

$$= \int_0^1 4z(y)_0^1\,dz.$$

$$= \int_0^1 4z\,dz$$

$$= 4\left(\frac{z^2}{2}\right)_0^1.$$

$$= 4/2.$$

$$= 2.$$

$$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds + \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = -\int_0^1\int_0^1 dx\,dz + 0.$$

$$= -\int_0^1 [x]_0^1\,dz.$$

$$= -\int_0^1 dz = -(z)_0^1.$$

$$= -1$$

$$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 y\,dx\,dy.$$

$$= \int_0^1 y\,[x]_0^1\,dy.$$

$$= \int_0^1 y\,dy.$$

$$= \left[\frac{y^2}{2}\right]_0^1.$$

$$= 1/2.$$

$$\int \vec{F}\cdot\hat{n}\,ds = 2 - 1 + 1/2.$$

$$= 3/2 \quad \text{——— (2)}.$$

from (1) & (2) LHS = RHS hence verified.

2 Verify gauss divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ where S is the surface of the cuboid formed by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

Now :

$$\nabla\cdot\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}x^2 + \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}z^2.$$

$$= 2x + 2y + 2z.$$

$$\nabla\cdot\vec{F} = 2(x+y+z)$$

RHS :

$$\iiint_V \nabla\cdot\vec{F}\,dv$$

$$= \int_0^c\int_0^b\int_0^a 2(x+y+z)\,dx\,dy\,dz.$$

$$= 2\int_0^c\int_0^b \left(\frac{x^2}{2} + yx + zx\right)_{x=0}^{a} dy\,dz.$$

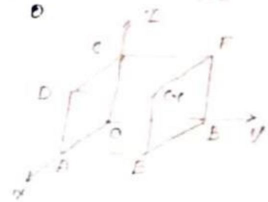
$$= 2\int_0^c\int_0^b \left[\frac{a^2}{2} + ay + az\right] dy\,dz.$$

$$= 2\int_0^c \left[\frac{a^2}{2}y + \frac{ay^2}{2} + ayz\right]_{y=0}^{b} dz$$

$$= 2\int_0^c \left[\frac{a^2 b}{2} + \frac{ab^2}{2} + abz\right] dz$$

$$= 2\left[\frac{a^2 b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$



$$= 2\frac{abc}{2}\,[a+b+c]$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = abc\,[a+b+c] \quad \text{—(1)}$$

Face	$\hat{n}$	$\vec{F}\cdot\hat{n}$	eqn	$\vec{F}\cdot\hat{n}$ on S	ds	$\iint \vec{F}\cdot\hat{n}\,ds$
AEGD	$\vec{i}$	x^2	$x = a$	a^2	$dy\,dz$	$\int_0^c\int_0^b a^2\,dy\,dz$
OBFC	$-\vec{i}$	$-x^2$	$x = 0$	0	$dy\,dz$	0
EBFG	$\vec{j}$	y^2	$y = b$	b^2	$dx\,dz$	$\int_0^c\int_0^a b^2\,dx\,dz$
OADC	$-\vec{j}$	$-y^2$	$y = 0$	0	$dx\,dz$	0
DGFC	$\vec{k}$	z^2	$z = c$	c^2	$dx\,dy$	$\int_0^b\int_0^a c^2\,dx\,dy$
OAEB	$-\vec{k}$	$-z^2$	$z = 0$	0	$dx\,dy$	0

$$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds + \iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^b a^2\,dy\,dz + 0$$

$$= a^2\int_0^c (y)_0^b\,dz$$

$$= a^2\int_0^c b\,dz$$

$$= a^2\left[b(z)_0^c\right]$$

$= a^2bc$

$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds + \iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^c\int_0^a b^2\,dx\,dz + 0.$

$= b^2\int_0^c (x)_0^a$

$= b^2\int_0^c a\,dz$

$= b^2\int_0^c (x)_0^a\,.\,dz$

$= b^2a\int_0^c dz.$

$= b^2a\,(z)_0^c$

$= b^2ac.$

$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds + \iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^b\int_0^a c^2\,dx\,dy\,. + 0$

$= c^2\int_0^b (x)_0^a\,dy.$

$= c^2\int_0^b a\,dy.$

$= c^2a\int_0^b dy$

$= c^2ab.$

$\iint_S \vec{F}\cdot\hat{n}\,ds = a^2bc + b^2ac + c^2ab.$

$= abc\,[a+b+c]$ —— (2)

from (1) & (2) LHS = RHS

Hence verified.