Unit - II
Vector calculus

Gradient :

Let $\phi(x, y, z)$ be a scalar point function and is continuously differentiable. Then the vector

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z} \text{ is called the gradient of}$$

the scalar fn. ϕ.

i.e., grad $\phi = \nabla \phi$

Problems

1] find $\nabla \phi$ where $\phi = x^2 + y^2 + z^2$

Soln.

$$\text{Grad. } \phi \text{ (or) } \nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \vec{j} \frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \vec{k} \frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$\nabla \phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

2]. find $\nabla \phi$ where $\overline{\phi} = 3x^2 y - y^3 z^2$ at $(1,1,1)$

Soln.

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i} \frac{\partial}{\partial x}(3x^2 y - y^3 z^2) + \vec{j} \frac{\partial}{\partial y}(3x^2 y - y^3 z^2) + \vec{k} \frac{\partial}{\partial z}(3x^2 y - y^3 z^2)$$

$$= \vec{i}[6xy - 0] + \vec{j}[3x^2 - 3y^2 z^2] + \vec{k}[0 - 2y^3 z]$$

$$\nabla \phi = 6xy\,\vec{i} + (3x^2 - 3y^2 z^2)\vec{j} - 2y^3 z\,\vec{k}$$

$$\nabla \phi_{(1,1,1)} = 6(1)(1)\vec{i} + (3-3)\vec{j} - 2(1)(1)\vec{k}$$

$$= 6\vec{i} + 0\vec{j} - 2\vec{k}$$

$$= 6\vec{i} - 2\vec{k}$$

UNIT-I   VECTOR CALCULUS   DERIVATIVES: Gradient of a scalar field, Directional Derivative

3. Find the maximum directional derivative of $\phi = xyz^2$ at $(1, 0, 3)$.

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz^2) + \vec{j}\frac{\partial}{\partial y}(xyz^2) + \vec{k}\frac{\partial}{\partial z}(xyz^2)$$

$$\nabla\phi = \vec{i}(yz^2) + \vec{j}(xz^2) + \vec{k}(xy\,2z)$$

$$\nabla\phi_{(1,0,3)} = \vec{i}(0) + \vec{j}(1)(9) + \vec{k}(0)$$

$$= 9\vec{j} \qquad \text{maximum D.D} = \sqrt{91} = 8$$

4. Find $\nabla\phi$ where $\phi = xyz$ at $(1, 2, 3)$

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial(xyz)}{\partial x} + \vec{j}\frac{\partial(xyz)}{\partial y} + \vec{k}\frac{\partial(xyz)}{\partial z}$$

$$\nabla\phi = \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$\nabla\phi_{(1,2,3)} = \vec{i}(2)(3) + \vec{j}(1)(3) + \vec{k}(1)(2)$$

$$= 6\vec{i} + 3\vec{j} + 2\vec{k}$$

5. If $\nabla\phi = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$, find ϕ.

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \vec{i}(yz) + \vec{j}(zx) + \vec{k}(xy)$$

Equating w.r.to $\vec{i}, \vec{j}, \vec{k}$

$\dfrac{\partial\phi}{\partial x} = yz$	$\dfrac{\partial\phi}{\partial y} = zx$	$\dfrac{\partial\phi}{\partial z} = xy$
Integrate w.r.to x	w.r.to y	w.r.to z
$\phi = xyz + f(y,z)$	$\phi = xyz + f(x,z)$	$\phi = xyz + f(x,y)$

In general,
$$\phi = xyz + c$$

UNIT-I   VECTOR CALCULUS   DERIVATIVES: Gradient of a scalar field, Directional Derivative

6J. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, such that $|\vec{r}| = r$,

Prove that

i). $\nabla r = \dfrac{\vec{r}}{r} = \hat{r}$

ii) $\nabla\left(\dfrac{1}{r}\right) = \dfrac{-\vec{r}}{r^3} = \dfrac{-\hat{r}}{r^2}$

iii). $\nabla r^n = n r^{n-2} \vec{r}$

iv). $\nabla f(r) = f'(r) \nabla r$

Soln.

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\Rightarrow r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow r^2 = x^2 + y^2 + z^2 \longrightarrow (1)$

Diff. (1) w.r. to x, y, z,

$2r \dfrac{\partial r}{\partial x} = 2x$ | $2r \dfrac{\partial r}{\partial y} = 2y$ | $2r \dfrac{\partial r}{\partial z} = 2z$

$\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ | $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ | $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

i). $\nabla r = \vec{i}\dfrac{\partial r}{\partial x} + \vec{j}\dfrac{\partial r}{\partial y} + \vec{k}\dfrac{\partial r}{\partial z}$

$= \vec{i}\left(\dfrac{x}{r}\right) + \vec{j}\left(\dfrac{y}{r}\right) + \vec{k}\left(\dfrac{z}{r}\right)$

$= \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$

$\nabla r = \dfrac{\vec{r}}{r}$

ii). $\nabla\left(\dfrac{1}{r}\right) = \vec{i}\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right) + \vec{j}\dfrac{\partial}{\partial y}\left(\dfrac{1}{r}\right) + \vec{k}\dfrac{\partial}{\partial z}\left(\dfrac{1}{r}\right)$

$= \vec{i}\left[-\dfrac{1}{r^2}\dfrac{\partial r}{\partial x}\right] + \vec{j}\left[-\dfrac{1}{r^2}\dfrac{\partial r}{\partial y}\right] + \vec{k}\left[-\dfrac{1}{r^2}\dfrac{\partial r}{\partial z}\right]$

$= \vec{i}\left[-\dfrac{1}{r^2}\times\dfrac{x}{r}\right] + \vec{j}\left[-\dfrac{1}{r^2}\times\dfrac{y}{r}\right] + \vec{k}\left[-\dfrac{1}{r^2}\times\dfrac{z}{r}\right]$

$= -\dfrac{1}{r^3}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$

$\nabla\left(\dfrac{1}{r}\right) = \dfrac{\vec{r}}{r^3}$

iii). $\nabla r^n = \vec{\imath}\, \dfrac{\partial (r^n)}{\partial x} + \vec{\jmath}\, \dfrac{\partial (r^n)}{\partial y} + \vec{k}\, \dfrac{\partial (r^n)}{\partial z}$

$= \vec{\imath}\, n r^{n-1} \dfrac{\partial r}{\partial x} + \vec{\jmath}\, n r^{n-1} \dfrac{\partial r}{\partial y} + \vec{k}\, n r^{n-1} \dfrac{\partial r}{\partial z}$

$= n r^{n-1} \left[\vec{\imath}\, \dfrac{\partial r}{\partial x} + \vec{\jmath}\, \dfrac{\partial r}{\partial y} + \vec{k}\, \dfrac{\partial r}{\partial z} \right]$

$= n r^{n-1} \left[\vec{\imath}\left(\dfrac{x}{r}\right) + \vec{\jmath}\left(\dfrac{y}{r}\right) + \vec{k}\left(\dfrac{z}{r}\right) \right]$

$= \dfrac{n r^{n-1}}{r} \left[x\vec{\imath} + y\vec{\jmath} + z\vec{k} \right]$

$= \dfrac{n r^{n-1}}{r}\, \vec{r}$

$\nabla r^n = n r^{n-2}\, \vec{r}$

iv). $\nabla f(r) = \vec{\imath}\, \dfrac{\partial}{\partial x} f(r) + \vec{\jmath}\, \dfrac{\partial}{\partial y} f(r) + \vec{k}\, \dfrac{\partial}{\partial z} f(r)$

$= \vec{\imath}\, f'(r) \dfrac{\partial r}{\partial x} + \vec{\jmath}\, f'(r) \dfrac{\partial r}{\partial y} + \vec{k}\, f'(r) \dfrac{\partial r}{\partial z}$

$= f'(r) \left[\vec{\imath}\left(\dfrac{x}{r}\right) + \vec{\jmath}\left(\dfrac{y}{r}\right) + \vec{k}\left(\dfrac{z}{r}\right) \right]$

$= \dfrac{f'(r)}{r} \left[x\vec{\imath} + y\vec{\jmath} + z\vec{k} \right]$

$= f'(r) \times \dfrac{\vec{r}}{r}$

$\nabla f(r) = f'(r)\, \nabla r \qquad (\text{from } \ddot{\imath})$

UNIT-I   VECTOR CALCULUS   DERIVATIVES: Gradient of a scalar field, Directional Derivative

Surfaces :

Unit normal vector $\hat{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$

Normal derivative $= |\nabla \phi|$

Directional derivative $= \nabla \phi \cdot \dfrac{\vec{a}}{|\vec{a}|}$

Angle between the Surfaces :

$$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| \, |\nabla \phi_2|}$$

If any two surfaces are cut orthogonally, then $\nabla \phi_1 \cdot \nabla \phi_2 = 0$

U. Find the unit normal to the surface $x^2 + xy + z^2 = 4$ at $(1, -1, 2)$.

Soln.

Let $\phi = x^2 + xy + z^2 - 4$

Unit normal vector $\hat{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$

Now $\nabla \phi = \vec{\imath} \dfrac{\partial \phi}{\partial x} + \vec{\jmath} \dfrac{\partial \phi}{\partial y} + \vec{k} \dfrac{\partial \phi}{\partial z}$

$\quad = \vec{\imath} \dfrac{\partial}{\partial \phi}(x^2 + xy + z^2 - 4) + \vec{\jmath} \dfrac{\partial}{\partial y}(x^2 + xy + z^2 - 4)$

$\qquad + \vec{k} \dfrac{\partial}{\partial z}(x^2 + xy + z^2 - 4)$

$\quad = \vec{\imath}(2x + y) + \vec{\jmath}(x) + \vec{k}(2z)$

$\nabla \phi_{(1,-1,2)} = \vec{\imath}(2(1) - 1) + \vec{\jmath}(1) + \vec{k}(2(2))$

$\quad = \vec{\imath} + \vec{\jmath} + 4\vec{k}$

$\therefore \hat{n} = \dfrac{\vec{\imath} + \vec{\jmath} + 4\vec{k}}{\sqrt{1 + 1 + 16}} = \dfrac{\vec{\imath} + \vec{\jmath} + 4\vec{k}}{\sqrt{18}}$

2]. Find the directional derivative of $\phi = xyz$ at the direction of $\vec{\imath} + \vec{\jmath} + \vec{k}$

UNIT-I   VECTOR CALCULUS   DERIVATIVES: Gradient of a scalar field, Directional Derivative

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$\nabla\phi_{(1,1,1)} = \vec{i}(1)(1) + \vec{j}(1)(1) + \vec{k}(1)(1)$$

$$= \vec{i} + \vec{j} + \vec{k}$$

Given $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$$|\vec{a}| = \sqrt{1+1+1} = \sqrt{3}$$

$$DD = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= (\vec{i}+\vec{j}+\vec{k}) \cdot \frac{(\vec{i}+\vec{j}+\vec{k})}{\sqrt{3}}$$

$$= \frac{1+1+1}{\sqrt{3}} = \frac{3}{\sqrt{3}}$$

$$DD = \sqrt{3}$$

3]. Find the directional derivative of $\phi = x^2 + 2xy$ at $(1,-1,3)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$

Soln.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(2x+2y) + \vec{j}(2x) + \vec{k}(0)$$

$$\nabla\phi = (2x+2y)\vec{i} + 2x\vec{j}$$

$$\nabla\phi_{(1,-1,3)} = (2(1)+2(-1))\vec{i} + 2(1)\vec{j}$$

$$= -2\vec{j}$$

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1+4+4}$$

$$= \sqrt{9} = 3$$

$$DD = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = -2\vec{j} \cdot \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$$

$$= -\frac{4}{3}$$

UNIT-I   VECTOR CALCULUS   DERIVATIVES: Gradient of a scalar field, Directional Derivative

4]. What is the greatest rate of increase of $u = x^2 + yz^2$ at $(1, -1, 3)$

Soln.

$$\nabla u = \vec{i} \frac{\partial u}{\partial x} + \vec{j} \frac{\partial u}{\partial y} + \vec{k} \frac{\partial u}{\partial z}$$

$$= \vec{i} (2x) + \vec{j} (z^2) + \vec{k} (2yz)$$

$$\nabla u = 2x \vec{i} + z^2 \vec{j} + 2yz \vec{k}$$

$$\nabla u_{(1,-1,3)} = 2\vec{i} + 9\vec{j} + 2(-1)(3)\vec{k}$$

$$= 2\vec{i} + 9\vec{j} - 6\vec{k}$$

∴ The greatest rate increase in the direction of y.

5]. Find the angle b/w the normals to the surface $xy = z^2$ at the points $(1, 4, 2)$ & $(-3, -3, 3)$

Soln.

Given $xy = z^2$

$$\phi = xy - z^2$$

$$\nabla \phi = \vec{i} \frac{\partial \phi}{\partial x} + \vec{j} \frac{\partial \phi}{\partial y} + \vec{k} \frac{\partial \phi}{\partial z}$$

$$= \vec{i} (y) + \vec{j} (x) + \vec{k} (-2z)$$

$$= y \vec{i} + x \vec{j} - 2z \vec{k}$$

$$\nabla \phi_1 {}_{(1, 4, 2)} = 4\vec{i} + \vec{j} - 4\vec{k}$$

$$|\nabla \phi_1| = \sqrt{16 + 1 + 16} = \sqrt{33}$$

and $\nabla \phi_2 {}_{(-3, -3, 3)} = -3\vec{i} - 3\vec{j} - 6\vec{k}$

$$|\nabla \phi_2| = \sqrt{9 + 9 + 36} = \sqrt{54} = 3\sqrt{6}$$

∴ $$\cos \theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| \, |\nabla \phi_2|}$$

$$= \frac{(4\vec{i} + \vec{j} - 4\vec{k}) \cdot (-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{33} \; 3\sqrt{6}}$$

$$= \frac{-12 - 3 + 24}{3\sqrt{11 \times 3 \times 3 \times 2}} = \frac{9}{3 \times 3\sqrt{22}} = \frac{1}{\sqrt{22}}$$

$$\Rightarrow \theta = \cos^{-1} \left(\frac{1}{\sqrt{22}} \right)$$

6). Find the angle between the Surfaces

$x^2 - y^2 - z^2 = 11$ and $xy + yz - zx = 18$ at $(6, 4, 3)$

Soln.

Let $\phi_1 = x^2 - y^2 - z^2 - 11$

$\nabla \phi_1 = \vec{i}\dfrac{\partial \phi_1}{\partial x} + \vec{j}\dfrac{\partial \phi_1}{\partial y} + \vec{k}\dfrac{\partial \phi_1}{\partial z}$

$= \vec{i}(2x) + \vec{j}(-2y) + \vec{k}(-2z)$

$\nabla \phi_1 {}_{(6,4,3)} = 12\vec{i} - 8\vec{j} - 6\vec{k} \Rightarrow |\nabla \phi_1| = \sqrt{144 + 64 + 36}$
$= \sqrt{244}$

and $\phi_2 = xy + yz - zx - 18$

$\nabla \phi_2 = \vec{i}\dfrac{\partial \phi_2}{\partial x} + \vec{j}\dfrac{\partial \phi_2}{\partial y} + \vec{k}\dfrac{\partial \phi_2}{\partial z}$

$= \vec{i}(y - z) + \vec{j}(x + z) + \vec{k}(y - x)$

$\nabla \phi_2 {}_{(6,4,3)} = \vec{i} + 9\vec{j} - 2\vec{k} \Rightarrow |\nabla \phi_2| = \sqrt{1 + 81 + 4}$
$= \sqrt{86}$

$\therefore \cos\theta = \dfrac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1||\nabla \phi_2|}$

$= \dfrac{(12\vec{i} - 8\vec{j} - 6\vec{k}) \cdot (\vec{i} + 9\vec{j} - 2\vec{k})}{\sqrt{244} \quad \sqrt{86}}$

$= \dfrac{12(1) - 8(9) - 6(-2)}{\sqrt{244} \sqrt{86}}$

$= \dfrac{-48}{2\sqrt{61}\sqrt{86}}$

$\cos\theta = \dfrac{-24}{\sqrt{5246}}$

$\theta = \cos^{-1}\left[\dfrac{-24}{\sqrt{5246}}\right]$

7). Find a and b, such that the surfaces
$ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut
orthogonally at $(1, -1, 2)$

UNIT-I VECTOR CALCULUS   DERIVATIVES: Gradient of a scalar field, Directional Derivative

Soln.

Let $\phi_1 = ax^2 - byz - (a+2)x \longrightarrow (1)$

$$\nabla\phi_1 = \vec{i}\,\frac{\partial\phi_1}{\partial x} + \vec{j}\,\frac{\partial\phi_1}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\nabla\phi_1 = \vec{i}\,[2ax - (a+2)] + \vec{j}\,[-bz] + \vec{k}\,[-by]$$

$$\nabla\phi_1{}_{(1,-1,2)} = \vec{i}\,[2a - a - 2] - 2b\vec{j} + b\vec{k}$$

$$= (a-2)\vec{i} - 2b\vec{j} + b\vec{k}$$

and $\phi_2 = 4x^2 y + z^3 - 4$

$$\nabla\phi_2 = 8xy\,\vec{i} + 4x^2\,\vec{j} + 3z^2\,\vec{k}$$

$$\nabla\phi_2{}_{(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Given two surfaces are cut orthogonally,

i.e, $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

$$[(a-2)\vec{i} - 2b\vec{j} + b\vec{k}] \cdot [-8\vec{i} + 4\vec{j} + 12\vec{k}] = 0$$

$$-8(a-2) - 8b + 12b = 0$$

$$-8a + 16 - 8b + 12b = 0$$

$$-2a + b + 4 = 0$$

i.e, $2a - b - 4 = 0 \longrightarrow (2)$

Since $(1,-1,2)$ lies on the surface using (1).

$\phi_1(x,y,z) = 0$

$a(1)^2 - b(-1)(2) = (a+2)(1)$

$a + 2b - a - 2 = 0$

$$2b = 2 \Rightarrow \boxed{b = 1}$$

$(2) \Rightarrow 2a - 1 - 4 = 0 \Rightarrow 2a = 5 \Rightarrow \boxed{a = 5/2}$