1. Solve $x^2 y'' + 2xy' = 0$

Soln

Given $(x^2 D^2 + 2xD)y = 0 \rightarrow (1)$

Take $x = e^z$

$z = \log x$

$xD = D'$

$x^2 D^2 = D'(D'-1) = D'^2 - D'$

Subs. the above in (1),

$[D'^2 - D' + 2D']y = 0$

$[D'^2 + D']y = 0$

AE $m^2 + m = 0$ $\quad D' \rightarrow m$

$m(m+1) = 0$

$m = 0, m = -1$

$CF = Ae^{0z} + Be^{-z}$

$= A + Be^{-z}$

$\therefore$ The soln. is, $y = CF = A + Be^{-\log x} = A + \frac{B}{x}$.

2. Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = x \sin(\log x)$

Soln.

Given $(x^2 D^2 - 3xD + 4)y = x \sin(\log x) \rightarrow (1)$

Take $x = e^z$

$z = \log x$.

$xD = D'$

$x^2 D^2 = D'(D'-1) = D'^2 - D'$

Subs. the above in (1)

$[D'^2 - D' - 3D' + 4]y = e^z \sin z$

$[D'^2 - 4D' + 4]y = e^z \sin z$

AE $m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$m = 2, 2$

$CF = (A + Bz)e^{2z}$

$CF = [A + B\log x]\,x^2$

$PI = \frac{1}{D'^2 - 4D' + 4}\, e^{z} \sin z$

$= e^{z} \frac{1}{(D'+1)^2 - 4(D'+1) + 4} \sin z$ $\quad D' \to D' + a = D' + 1$

$= e^{z} \frac{1}{D'^2 + 1 + 2D' - 4D' - 4 + 4} \sin z$

$= e^{z} \frac{1}{D'^2 - 2D' + 1} \sin z$

$= e^{z} \frac{1}{-1 - 2D' + 1} \sin z$ $\quad D'^2 \to -b^2 = -1^2 = -1$

$= e^{z} \frac{1}{-2D'} \sin z$

$= \frac{e^{z}}{-2}\left[\frac{1}{D'} \sin z\right]$

$= \frac{e^{z}}{-2}[-\cos z]$

$PI = \frac{e^{z}\cos z}{2} \Rightarrow PI = \frac{x \log(\log x)}{2}$

The soln. is,

$y = CF + PI$

$y = [A + B\log x]\,x^2 + \frac{x\cos(\log x)}{2}$