



Gauss Divergence theorem :

The surface integral of normal component of vector function  $F$  over a closed surface  $S$  enclosing volume  $V$  is equal to the volume integral of divergence of  $F$  taking through out the volume  $V$

$$\text{i.e } \iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

Verify the Gauss divergence theorem (GDT) for  $\vec{F} = xz \vec{i} - y^2 \vec{j} + yz \vec{k}$  over the cube bounded by  $x=0, x=1, y=0, y=1, z=0, z=1$



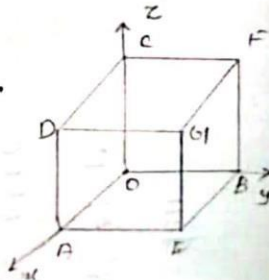


$$\iint_S \vec{F} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \vec{F} \, dV$$
$$\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$$
$$\nabla \cdot \vec{F} = \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (4xz\vec{i} - y^2\vec{j} + yz\vec{k})$$
$$= \frac{\partial}{\partial x} (4xz) + \frac{\partial}{\partial y} (-y^2) + \frac{\partial}{\partial z} (yz)$$
$$= 4z - 2y + y = 4z - y.$$
$$\nabla \cdot \vec{F} = 4z - y.$$

RHS:

$$\iiint_V \nabla \cdot \vec{F} \, dV = \int_0^1 \int_0^1 \int_0^1 (4z - y) \, dx \, dy \, dz.$$
$$= \int_0^1 \int_0^1 (4z - y) \, dy \, dz.$$
$$= \int_0^1 \left[ 4zy - \frac{y^2}{2} \right]_{y=0}^1 \, dz.$$
$$= \int_0^1 (4z - 1/2) \, dz.$$
$$= \left[ \frac{4z^2}{2} - \frac{1}{2}z \right]_{z=0}^1.$$
$$= \frac{4}{2} - \frac{1}{2}.$$

$$\iiint_V \nabla \cdot \vec{F} \, dV = 3/2 \quad \text{--- (1)}$$



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Surface	Equation	Normal Vector $\hat{n}$	Dot Product $\vec{F} \cdot \hat{n}$	Area Element $ds$	Equation	Value of $\vec{F} \cdot \hat{n}$ on $S$	Integral $\iint_S \vec{F} \cdot \hat{n} ds$
$S_1$ AEGD	$x=1$	$\hat{i}$	$4xz$	$dydz$	$x=1$	$4z$	$\int_0^1 \int_0^1 4z dy dz$
$S_2$ OBFC	$x=0$	$-\hat{i}$	$-4xz$	$dydz$	$x=0$	$0$	$0$
$S_3$ EFGI	$y=1$	$\hat{j}$	$-y^2$	$dx dz$	$y=1$	$-1$	$\int_0^1 \int_0^1 (-1) dx dz$
$S_4$ OADC	$y=0$	$-\hat{j}$	$+y^2$	$dx dz$	$y=0$	$0$	$0$
$S_5$ DGFC	$z=1$	$\hat{k}$	$yz$	$dx dy$	$z=1$	$y$	$\int_0^1 \int_0^1 y dx dy$
$S_6$ OAFB	$z=0$	$-\hat{k}$	$-yz$	$dx dy$	$z=0$	$0$	$0$



$$\Rightarrow \iint_V \vec{F} \cdot \hat{n} \, ds = \iint_{S_1} \dots + \iint_{S_2} \dots + \iint_{S_3} \dots + \iint_{S_4} \dots + \iint_{S_5} \dots + \iint_{S_6} \dots$$

$$\begin{aligned} \iint_{S_1} \vec{F} \cdot \hat{n} \, ds + \iint_{S_2} \vec{F} \cdot \hat{n} \, ds &= \int_0^1 \int_0^1 4z \, dy \, dz + 0 \\ &= \int_0^1 4z(y)_0^1 \, dz \\ &= \int_0^1 4z \, dz \\ &= 4 \left( \frac{z^2}{2} \right)_0^1 \\ &= 4/2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \iint_{S_3} \vec{F} \cdot \hat{n} \, ds + \iint_{S_4} \vec{F} \cdot \hat{n} \, ds &= - \int_0^1 \int_0^1 dx \, dz + 0 \\ &= - \int_0^1 [x]_0^1 \, dz \\ &= - \int_0^1 dz = - (z)_0^1 \\ &= -1 \end{aligned}$$

$$\begin{aligned} \iint_{S_5} \vec{F} \cdot \hat{n} \, ds + \iint_{S_6} \vec{F} \cdot \hat{n} \, ds &= \int_0^1 \int_0^1 y \, dx \, dy \\ &= \int_0^1 [xy]_0^1 \, dy \\ &= \int_0^1 y \, dy \\ &= \left[ \frac{y^2}{2} \right]_0^1 \\ &= 1/2 \end{aligned}$$

$$\begin{aligned} \iint \vec{F} \cdot \hat{n} \, ds &= 2 - 1 + 1/2 \\ &= 3/2 \quad \text{--- (5)} \end{aligned}$$

from (4) & (5) LHS = RHS hence verified.





2. Verify Gauss divergence theorem for  $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$  where  $V$  is the surface of the cuboid formed by the planes  $x=0, x=a, y=0, y=b, z=0, z=c$ .

$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

Now :

$$\begin{aligned} \nabla \cdot \vec{F} &= \left( \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (x^2\vec{i} + y^2\vec{j} + z^2\vec{k}) \\ &= \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial z} z^2 \\ &= 2x + 2y + 2z. \end{aligned}$$

$$\nabla \cdot \vec{F} = 2(x+y+z)$$

RHS :

$$\iiint_V \nabla \cdot \vec{F} \, dv$$

$$= \int_0^c \int_0^b \int_0^a 2(x+y+z) \, dx \, dy \, dz.$$

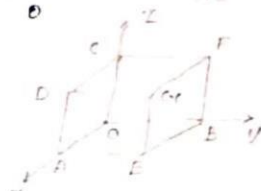
$$= 2 \int_0^c \int_0^b \left( \frac{x^2}{2} + yx + zx \right) \Big|_{x=0}^a \, dy \, dz.$$

$$= 2 \int_0^c \int_0^b \left[ \frac{a^2}{2} + ay + az \right] \, dy \, dz.$$

$$= 2 \int_0^c \left[ \frac{a^2}{2}y + \frac{ay^2}{2} + ayz \right] \Big|_{y=0}^b \, dz$$





$$\begin{aligned}
 &= 2 \int_0^c \left[ \frac{a^2 b}{2} + \frac{ab^2}{2} + abx \right] dx \\
 &= 2 \left[ \frac{a^2 b}{2} x + \frac{ab^2}{2} x + ab \frac{x^2}{2} \right]_0^c \\
 &= 2 \left[ \frac{a^2 bc}{2} + \frac{ab^2 c}{2} + \frac{abc^2}{2} \right] \\
 &= 2 \frac{abc}{2} [a+b+c] \\
 \iiint \nabla \cdot \vec{F} \, dV &= abc [a+b+c] \quad \text{--- (1)}
 \end{aligned}$$


Face	$\hat{n}$	$\vec{F} \cdot \hat{n}$	eqn	$\frac{\vec{F} \cdot \hat{n}}{dV}$	$ds$	$\iint \vec{F} \cdot \hat{n} \, ds$
AEFD	$\vec{i}$	$x^2$	$x = a$	$a^2$	$dy \, dz$	$\int_0^c \int_0^b a^2 \, dy \, dz$
OBFC	$-\vec{i}$	$-x^2$	$x = 0$	0	$dy \, dz$	0
EBFG	$\vec{j}$	$y^2$	$y = b$	$b^2$	$dx \, dz$	$\int_0^c \int_0^a b^2 \, dx \, dz$
ODAC	$-\vec{j}$	$-y^2$	$y = 0$	0	$dx \, dz$	0
DOFC	$\vec{k}$	$z^2$	$z = c$	$c^2$	$dx \, dy$	$\int_0^b \int_0^a c^2 \, dx \, dy$
OAEB	$-\vec{k}$	$-z^2$	$z = 0$	0	$dx \, dy$	0

$$\begin{aligned}
 \iint_{S_1} \vec{F} \cdot \hat{n} \, ds + \iint_{S_2} \vec{F} \cdot \hat{n} \, ds &= \int_0^c \int_0^b a^2 \, dy \, dz + 0 \\
 &= a^2 \int_0^c (y)_0^b \, dz \\
 &= a^2 \int_0^c b \, dz \\
 &= a^2 \int_0^c b(z)_0^c \, dz
 \end{aligned}$$



$$= a^2bc$$

$$\iint_{S_3} \vec{F} \cdot \hat{n} \, ds + \iint_{S_H} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^a \int_0^b b^2 \, dz \, dx \, dy + 0$$

$$= b^2 \int_0^c (x) \Big|_0^a \, dz$$

$$= b^2 \int_0^c ax \, dz$$

$$= b^2 \int_0^c (ax) \Big|_0^a \, dz$$

$$= b^2 a \int_0^c dz$$

$$= b^2 a (z) \Big|_0^c$$

$$= b^2 ac$$

$$\iint_{S_5} \vec{F} \cdot \hat{n} \, ds + \iint_{S_6} \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_0^a c^2 \, dz \, dy + 0$$

$$= c^2 \int_0^b (z) \Big|_0^a \, dy$$

$$= c^2 \int_0^b a \, dy$$

$$= c^2 a \int_0^b dy$$

$$= c^2 ab$$

$$\iint_S \vec{F} \cdot \hat{n} \, ds = a^2bc + b^2ac + c^2ab$$

$$= abc [a + b + c] \quad \text{--- (2)}$$

from (1) + (2) LHS = RHS  
Hence verified.