Solenoidal & Irrotational vector:

Solenoidal vector: $\nabla \cdot \vec{F} = 0$

Irrotational Vector: $\nabla \times \vec{F} = \vec{0}$

Problems

1] Find 'a' such that $(3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$ is solenoidal.

Soln.

Given $\vec{F} = (3x-2y+z)\vec{i} + (4x+ay-z)\vec{j} + (x-y+2z)\vec{k}$

and $\nabla \cdot \vec{F} = 0$

$$\frac{\partial}{\partial x}(3x-2y+z) + \frac{\partial}{\partial y}(4x+ay-z) + \frac{\partial}{\partial z}(x-y+2z) = 0$$

$$3+a+2 = 0$$

$$\boxed{a = -5}$$

2] Show that $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

Soln.

Given $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(z\vec{i} + x\vec{j} + y\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$\nabla \cdot \vec{F} = 0$

$\therefore \vec{F}$ is solenoidal

3] Show that $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ is irrotational.

Soln.

Given $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$

to prove $\nabla \times \vec{F} = \vec{0}$

Now,

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$= \vec{i}[x - x] - \vec{j}[y - y] + \vec{k}[z - z]$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

4]. If $\vec{A}$ and $\vec{B}$ are irrotational, prove that $\vec{A} \times \vec{B}$ is solenoidal.

Soln.

Given $\vec{A}$ & $\vec{B}$ are irrotational.

ie., $\nabla \times \vec{A} = \vec{0}$ and $\nabla \times \vec{B} = \vec{0}$

WKT $\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - (\nabla \times \vec{B}) \cdot \vec{A}$

$$= \vec{0} \cdot \vec{B} - \vec{0} \cdot \vec{A}$$

$$= 0 - 0$$

$$= 0$$

Hence $\vec{A} \times \vec{B}$ is solenoidal.

5]. Find the values of a, b, c so that the vector $\vec{F} = (x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$ may be irrotational.

Soln.

Given $\vec{F}$ is irrotational.

ie., $\nabla \times \vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & -x+cy+2z \end{vmatrix} = \vec{0}$$

$\vec{i}(c+1) - \vec{j}(-1-a) + \vec{k}(b-1) = 0\vec{i} + 0\vec{j} + 0\vec{k}$

Equating the coefficients of like terms,

$c+1=0, \quad 1+a=0, \quad b-1=0$

$c=-1 \quad a=-1 \quad b=1$

6]. Show that $\vec{F} = (y^2+2xz^2)\vec{i} + (2xy-z)\vec{j} + (2x^2z - y + 2z)\vec{k}$ is irrotational and hence find its scalar potential.

Soln.

Given $\vec{F} = (y^2+2xz^2)\vec{i} + (2xy-z)\vec{j} + (2x^2z - y + 2z)\vec{k}$

Now

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+2xz^2 & 2xy-z & 2x^2z-y+2z \end{vmatrix}$$

$$= \vec{i}[-1+1] - \vec{j}[4xz-4xz] + \vec{k}[2y-2y]$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

$$\nabla \times \vec{F} = \vec{0}$$

Hence $\vec{F}$ is irrotational.

$$\Rightarrow \vec{F} = \nabla \phi$$

$$(y^2+2xz^2)\vec{i} + (2xy-z)\vec{j} + (2x^2z-y+2z)\vec{k} = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

Equating the coefficients of $\vec{i}, \vec{j}$ & $\vec{k}$, we get

$\frac{\partial\phi}{\partial x} = y^2+2xz^2 \;\Big|\; \frac{\partial\phi}{\partial y} = 2xy+z \;\Big|\; \frac{\partial\phi}{\partial z} = 2x^2z-y+2z$

Integrating partially w.r. to x, y, z,

$\phi = xy^2 + x^2z^2 + f_1(y,z) \longrightarrow (1)$

$\phi = xy^2 - yz + f_2(x,z) \longrightarrow (2)$

$\phi = x^2z^2 - yz + z^2 + f_3(x,y) \longrightarrow (3)$

Comparing (1), (2), and (3), we get

$\phi = xy^2 - yz + x^2z^2 + z^2 + c$, where c is the arbitrary constant