POISSON DISTRIBUTION: (DISCRETE)

A random variable x is said to follow poisson distribution if it assumes only non-negative values and its probability mass function is given by.

$$p(X=x) = p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots, \infty$$

where λ is known as the parameter of the poisson distribution.

Examples:

(1) No. of mistakes committed by a typist per page.
(2) No. of defective items produced in the factory.

Note:

The Binomial distribution tends to poisson distribution when

(i) The number of trials is indefinitely large (ie) $n \to \infty$
(ii) The probability of success is very small. (ie) $p \to 0$

Find the Moment generating function, its Mean and Variance using poisson distribution.

Moment generating function:

The MGF of a poisson distribution is

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(X=x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} \qquad \left[\sum_{x=0}^{\infty} \frac{a^x}{x!} = e^a\right]$$

$$= e^{\lambda(e^t-1)}$$

$$\therefore M_x(t) = e^{\lambda(e^t-1)}$$

Mean:

$$E(x) = \left\{\frac{d}{dt}(M_x(t))\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left[e^{\lambda(e^t-1)}\right]\right\}_{t=0}$$

$$= \left[e^{\lambda(e^t-1)} \cdot \lambda e^t\right]_{t=0}$$

$$= e^{\lambda(e^0-1)} \lambda e^0 = \lambda$$

$$\therefore E(x) = \lambda$$

Now $E(x^2) = \left\{\frac{d^2}{dt^2}\left[M_x(t)\right]\right\}_{t=0}$

$$= \left\{\frac{d^2}{dt^2}\left[e^{\lambda(e^t-1)}\right]\right\}_{t=0}$$

$$= \left\{\frac{d}{dt}\left[e^{\lambda(e^t-1)}\lambda e^t\right]\right\}_{t=0}$$

$$= \left\{\lambda\left[e^{\lambda(e^t-1)}e^t + e^t e^{\lambda(e^t-1)} \cdot \lambda e^t\right]\right\}_{t=0}$$

$$= \lambda\left[e^{\lambda(e^0-1)} \cdot e^0 + e^0 e^{\lambda(e^0-1)} \cdot \lambda e^0\right]$$

$$= \lambda [1 + \lambda]$$
$$= \lambda + \lambda^2$$

$\therefore$ Variance:

$$Var(x) = E(x^2) - [E(x)]^2$$
$$= \lambda + \lambda^2 - \lambda^2$$
$$= \lambda$$

PROBLEMS:

(1) If the moment generating function of the RV is $e^{4(e^t - 1)}$ find $p(x = \mu + \sigma)$ where μ and σ^2 are the mean and variance of the poisson.

Soln:

WKT MGF of poisson distribution is

$$M_x(t) = e^{\lambda(e^t - 1)}$$

Given: $M_x(t) = e^{4(e^t - 1)}$

$$\Rightarrow \lambda = 4$$

Mean (λ): $\mu = 4$

Variance (λ): $\sigma^2 = 4$

Now, $\sigma = \sqrt{\text{variance}} = \sqrt{4} = 2$

$$\therefore p(x = \mu + \sigma) = p(x = 4+2)$$
$$= p(x=6)$$
$$= \frac{e^{-4} 4^6}{6!}$$

(2) If x is a poisson variate $p(x=2) = 9p(x=4) + 90p(x=6)$

find (i) Mean of x, (ii) Variance of x

Soln:
WKT $p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!}, \ x = 0, 1, \ldots, \infty$

Given: $p(x=2) = 9p(x=4) + 90\,p(x=6)$

$$\Rightarrow \frac{e^{-\lambda}\lambda^2}{2!} = \frac{9 \cdot e^{-\lambda}\lambda^4}{4!} + \frac{90 \cdot e^{-\lambda}\lambda^6}{6!}$$

$$\Rightarrow \frac{\cancel{e^{-\lambda}}\,\cancel{\lambda^2}}{\cancel{2!}} = \frac{\overset{3}{\cancel{9}}\,\cancel{e^{-\lambda}}\,\cancel{\lambda^4}^{2}}{\underset{4\times\cancel{3}}{\cancel{4!}}}\left[1 + \overset{2}{\cancel{10}}\frac{\lambda^2}{\underset{3}{\cancel{6}}\times\cancel{5}}\right]$$

$$\Rightarrow 1 = \frac{3}{4}\lambda^2\left[1+\frac{\lambda^2}{3}\right]$$

$$\Rightarrow 3\lambda^2+\lambda^4 = 4$$

$$\Rightarrow \lambda^4+3\lambda^2-4 = 0$$

$$\Rightarrow \lambda^2 = \frac{-3\pm\sqrt{9+16}}{2} = \frac{-3\pm 5}{2} = 1, -4$$

$$\Rightarrow \lambda^2 = 1 \ ; \ \lambda^2 = -4$$

$$\Rightarrow \lambda = \pm 1 \text{ (or) } \lambda = \pm 2i$$

$$\Rightarrow \lambda = 1 \quad [\because \lambda > 0]$$

$\therefore$ Mean of $x = \lambda = 1$

& Variance of $x = \lambda = 1$

(3) If x and y are independent poisson variate such that $p(x=1) = p(x=2)$ and $p(y=2) = p(y=3)$. Find the variance of $x-2y$.

Soln: WKT $p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots \infty$

Given: $p(x=1) = p(x=2)$

Given: $p(x=1) = p(x=2)$

$$\frac{e^{-\lambda}\lambda^{1}}{1!} = \frac{e^{-\lambda}\lambda^{2}}{2!}$$

$$\Rightarrow \lambda = 2$$

Also given: $p(y=2) = p(y=3)$

$$\frac{e^{-\mu}\mu^{2}}{2!} = \frac{e^{-\mu}\mu^{3}}{3!}$$

$$\Rightarrow \mu = 3$$

$\therefore$ $var(x) = \lambda = 2$ and $var(y) = \mu = 3$

Now $var(x-2y) = 1^2 var(x) + 2^2 var(y)$

$$= 2 + 4(3) = 14$$

(4) If x is a poisson variate with $p(x=1) = 3/10$ and $p(x=2) = 1/5$. Find $p(x=0)$, $p(x=3)$.

Soln:

Given: $p(x=1) = 3/10$;

$p(x=2) = 1/5$

WKT $p(x=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, x = 0, 1, \ldots, \infty$

$\Rightarrow \frac{e^{-\lambda}\lambda}{1!} = 3/10$ —— ①

$\frac{e^{-\lambda}\lambda^2}{2!} = 1/5$ —— ②

Now ② ÷ ①

$$\Rightarrow \frac{\frac{\cancel{e^{-\lambda}}\lambda^{\cancel{2}}}{2!}}{\frac{\cancel{e^{-\lambda}}\cancel{\lambda}}{1!}} = \frac{3/\cancel{10}\,2}{1/\cancel{5}}$$

$$\Rightarrow \frac{\lambda}{2} = \frac{2}{3}$$

$$\Rightarrow \lambda = 4/3$$

$$\therefore p(x=0) = \frac{e^{-4/3}(4/3)^0}{0!}$$

$$p(x=3) = \frac{e^{-4/3}(4/3)^3}{3!}$$

(5) A car hire firm has two cars which it hires out day by day. The no. of demands for a car on each day is distributed as poisson variate with mean 1.5. Calculate the proportion of days on which
(i) Neither car is used.
(ii) Some demand is refused.

Soln: Given: $\lambda = 1.5$

WKT poisson distribution is $p(x=x) = \dfrac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, \ldots \infty$

$$= \frac{e^{-1.5}(1.5)^x}{x!}$$

(i) p(neither car is used) = p(x=0)

$$= \frac{e^{-1.5}(1.5)^0}{0!} = e^{-1.5}$$

(ii) p(Some demand is refused) =

p(the no. of demands to be more than 2)

$$= p(x>2)$$

$$= 1 - p(x \le 2)$$

$$= 1 - [p(x=0) + p(x=1) + p(x=2)]$$

$$= 1 - \left[\frac{e^{-1.5}(1.5)^0}{0!} + \frac{e^{-1.5}(1.5)^1}{1!} + \frac{e^{-1.5}(1.5)^2}{2!}\right]$$

$$= 1 - e^{-1.5}\left[1 + 1.5 + \frac{(1.5)^2}{2}\right]$$

(6) The number of monthly breakdown of a computer is a R.v. having a poisson distribution with mean equal to 1.8. Find the probability that this computer will function for month with (i) Only one breakdown
(ii) no breakdown
(iii) two breakdown.

Soln: Given: $\lambda = 1.8$

WKT poisson distribution $p(x=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, \dots, \infty$

$$= \frac{e^{-1.8}(1.8)^{x}}{x!}$$

(i) p (only one breakdown) $= p(x=1)$

$$= \frac{e^{-1.8}(1.8)^{1}}{1!}$$

$$= e^{-1.8}(1.8)$$

(ii) p(No breakdown) $= p(x=0)$

$$= \frac{e^{-1.8}(1.8)^0}{0!} = e^{-1.8}$$

(iii) p(two breakdown) $= p(x=2)$

$$= \frac{e^{-1.8}(1.8)^2}{2!}$$

(7) Using a poisson distribution, find the probability that the ace of spades will be drawn from a pack of well shuffled cards atleast once in 104 Consecutive trials.

Soln:

Probability of the ace of spades $= \frac{1}{52}$

$n = 104$

$\therefore$ Mean $\lambda = n \times p = 104 \times \frac{1}{52}$

$= 2$

$\therefore$ poisson distribution $p(x=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}, x=0,1,\dots\infty$

$= \dfrac{e^{-2}2^{x}}{x!}$

p(Atleast once in 104 trials) $= p(x \geq 1)$

$= 1 - p(x<1)$

$= 1 - p(x=0)$

$= 1 - \dfrac{e^{-2}2^{0}}{0!} = 1 - e^{-2}$

(8) Six coins are tossed 6400 times. Using the poisson distribution, what is the approximate probability of getting six heads 10 times.

Soln: Given: $n = 6400$

Probability of getting one head with one coin $= \frac{1}{2}$

$\therefore$ probability of getting six heads with six coins $= \left(\frac{1}{2}\right)^{6}$

$= \frac{1}{64}$

$\therefore$ Mean: $\lambda = n \times p$

$= 6400 \times \frac{1}{64} = 100$

WKT poisson distribution $p(X=x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}, x = 0, 1, \ldots, \infty$

$$= \frac{e^{-100}\,100^{x}}{x!}$$

$\therefore$ P(getting six heads in 10 times) $= \dfrac{e^{-100}(100)^{10}}{10!}$

(9) Find the probability that atmost 5 defective fuses will be found in a box of 200 fuses if experiences shows that 2% of such fuses are defective.

(10) A manufacture of cotterpins knows that 5% of his product is defective. If he sells cotterpins in boxes of 100 and guarantees that not more than 10 pins will be defective. What is the approximate probability that a box will fail to meet the guaranteed quality?